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A regularization algorithm for zero points of accretive operators

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Abstract

A regularization algorithm with a computational error for treating accretive operators is investigated. A strong convergence theorem for zero points of accretive operators is established in a reflexive Banach space.

Keywords: accretive operator; fixed point; nonexpansive mapping; regularization algorithm; zero point

1 Introduction

In this paper, we are concerned with the problem of finding zero points of a mapping $A : E \to 2^{E^*}$; that is, finding a point x in the domain of A such that $0 \in Ax$. The domain of a mapping A is defined by the set $\{x \in E : Ax \neq 0\}$. Many important problems have reformulations which require finding zero points, for instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and optimization problems. It is well known that minimizing a convex function f can be reduced to finding zero points of the subdifferential mapping $A = \partial f$. One of the most popular techniques for solving the inclusion problem goes back to the work of Browder [1]. One of the basic ideas in the case of a Hilbert space H is reducing the above inclusion problem to a fixed point problem of the operator R_A defined by $R_A = (I + A)^{-1}$, which is called the classical resolvent of A. If A has some monotonicity conditions, the classical resolvent of A is with full domain and firmly nonexpansive, that is, $\|R_Ax - R_Ay\|^2 \leq \langle R_Ax - R_Ay, x - y\rangle$, $\forall x, y \in H$. The property of the resolvent ensures that the Picard iterative algorithm $x_{n+1} = R_Ax_n$ converges weakly to a fixed point of R_A, which is necessarily a zero point of A. Rockafellar introduced this iteration method and called it the proximal point algorithm; for more detail, see [2–4] and the references therein. Methods for finding zero points of monotone mappings in the framework of Hilbert spaces are based on the good properties of the resolvent R_A, but these properties are not available in the framework of Banach spaces.

In this paper, we study a viscosity algorithm with a computational error. A strong convergence theorem for zero points of accretive operators is established in a reflexive Banach space. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a strong convergence theorem is established in a reflexive Banach space. Two applications of the main results are also discussed in this section.

2 Preliminaries

In what follows, we always assume that E is a Banach space with the dual E^*. Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* or is said to have a *Gâteaux differentiable norm* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. E is said to be *uniformly smooth* or is said to have a *uniformly Fréchet differentiable norm* if the limit is attained uniformly for $x, y \in U_E$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$, $\forall x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping j is single-valued and uniformly norm to weak* continuous on each bounded subset of E.

Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be *α-contractive* if there exits a constant $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \le \alpha\|x - y\|$, $\forall x, y \in C$. T is said to be *nonexpansive* if $\alpha = 1$. T is said to be *pseudocontractive* if there exists some $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2$, $\forall x, y \in C$.

Recall that a closed convex subset C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, *i.e.*, $\sup\{\|x - y\| : y \in K\} < d(K)$, where $d(K)$ is the diameter of K. Let D be a nonempty subset of C. Let $Q : C \to D$. Q is said to be *contraction* if $Q^2 = Q$; *sunny* if for each $x \in C$ and $t \in (0, 1)$, we have $Q(tx + (1 - t)Qx) = Qx$; *sunny nonexpansive retraction* if Q is sunny, nonexpansive, and contraction. K is said to be a *nonexpansive retract* of C if there exists a nonexpansive retraction from C onto D; for more details, see [5] and the references therein.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \ge 0$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the *resolvent* of A.

One of classical methods of studying the problem $0 \in Ax$, where $A \subset E \times E$ is an accretive operator, is the proximal point algorithm (PPA) which was initiated by Martinet [6] and further developed by Rockafellar [3]. It is known that PPA is only weakly convergent; see Güler [7]. In many disciplines, including economics, image recovery, quantum physics, and control theory, problems arise in infinite dimension spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. The importance of strong convergence is also underlined in [7], where a convex function f is minimized via the proximal-point algorithm: it is shown that the rate of convergence of the value sequence $\{f(x_n)\}$ is better when $\{x_n\}$ converges strongly than when it converges weakly. Such properties have a direct impact when the process is executed directly in the underlying infinite dimensional space.

Regularization methods recently have been investigated for treating zero points of accretive operators; see [8–22] and the references therein. In this paper, zero points of m-accretive operators are investigated based on a viscosity iterative algorithm with a computational error. A strong convergence theorem for zero points of m-accretive operators is established in a reflexive Banach space.

In order to state our main results, we need the following lemmas.

Lemma 2.1 [23] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E. Let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$, $\forall n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2 [21] *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with a fixed point, and let $f : C \to C$ be a fixed contraction with the coefficient $\alpha \in (0,1)$. Let $\{x_t\}$ be a sequence generated by the following $x_t = tf(x_t) + (1-t)Tx_t$, where $t \in (0,1)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point x^* of T, which is the unique solution in $F(T)$ to the following variational inequality $\langle f(x^*) - x^*, j(x^* - p)\rangle \geq 0$, $\forall p \in F(T)$.*

Lemma 2.3 [24] *Let E be a Banach space, and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda x)$, where $J_\lambda = (I+\lambda A)^{-1}$ and $J_\mu = (I+\mu A)^{-1}$.*

Lemma 2.4 [25] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n$, $\forall n \geq 0$, where $\{t_n\}$ is a number sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^\infty t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^\infty c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3 Main results

Theorem 3.1 *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm, and let A be an m-accretive operator in E. Assume that $C := \overline{D(A)}$ is convex and has the normal structure. Let $f : C \to C$ be a fixed α-contraction. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)J_{r_n}\bigl(\alpha_n f(x_n) + (1-\alpha_n)x_n + e_{n+1}\bigr), \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{e_n\}$ is a sequence in E, $\{r_n\}$ is a positive real number sequence, and $J_{r_n} = (I + r_n A)^{-1}$. Assume that $A^{-1}(0)$ is not empty and the above control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\sum_{n=1}^\infty \|e_n\| < \infty$;
(d) $r_n \geq r > 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in A^{-1}(0)$, which is the unique solution to the following variational inequality $\langle f(\bar{x}) - \bar{x}, j(p - \bar{x})\rangle \leq 0$, $\forall p \in A^{-1}(0)$.

Proof Fixing $p \in A^{-1}(0)$, we find that

$$\begin{aligned}\|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)\bigl\|J_{r_n}\bigl(\alpha_n f(x_n) + (1-\alpha_n)x_n + e_{n+1}\bigr) - p\bigr\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\bigl(\alpha_n\bigl\|f(x_n) - p\bigr\| + (1-\alpha_n)\|x_n - p\| + \|e_{n+1}\|\bigr)\end{aligned}$$

$$
\begin{aligned}
&\le \bigl(1-\alpha_n(1-\beta_n)(1-\alpha)\bigr)\|x_n-p\| + \alpha_n(1-\beta_n)\bigl\|f(p)-p\bigr\| + \|e_{n+1}\| \\
&\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\} + \|e_{n+1}\| \\
&\ \ \vdots \\
&\le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\} + \sum_{i=1}^{n+1}\|e_i\| \\
&\le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\} + \sum_{i=1}^{\infty}\|e_i\| < \infty.
\end{aligned}
$$

This proves that the sequence $\{x_n\}$ is bounded. Put $y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n + e_{n+1}$. It follows that

$$
\begin{aligned}
\|y_{n+1}-y_n\| &\le \alpha_n\bigl\|f(x_{n+1})-f(x_n)\bigr\| + |\alpha_{n+1}-\alpha_n|\bigl\|f(x_n)-x_n\bigr\| \\
&\quad + (1-\alpha_{n+1})\|x_{n+1}-x_n\| + \|e_{n+1}\| + \|e_{n+1}\| \\
&\le \|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\bigl\|f(x_n)-x_n\bigr\| + \|e_{n+1}\| + \|e_{n+1}\|.
\end{aligned} \tag{3.1}
$$

In view of Lemma 2.3, we find that

$$
\begin{aligned}
\|J_{r_{n+1}}y_{n+1} - J_{r_n}y_n\| &= \left\|J_{r_n}\left(\frac{r_n}{r_{n+1}}y_{n+1} + \left(1-\frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}y_{n+1}\right) - J_{r_n}y_n\right\| \\
&\le \left\|\left(\frac{r_n}{r_{n+1}}y_{n+1} + \left(1-\frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}y_{n+1}\right) - y_n\right\| \\
&\le \|y_{n+1}-y_n\| + \frac{r_{n+1}-r_n}{r}M,
\end{aligned} \tag{3.2}
$$

where M is an appropriate constant such that $M \ge \sup_{n\ge 0}\{\|J_{r_{n+1}}y_{n+1} - y_{n+1}\|\}$. Substituting (3.1) into (3.2), we find that

$$
\begin{aligned}
&\|J_{r_{n+1}}y_{n+1} - J_{r_n}y_n\| - \|x_{n+1}-x_n\| \\
&\quad \le |\alpha_{n+1}-\alpha_n|\bigl\|f(x_n)-x_n\bigr\| + \|e_{n+1}\| + \|e_{n+1}\| + \frac{r_{n+1}-r_n}{r}M.
\end{aligned}
$$

In view of the restrictions (a), (c) and (d), we find that

$$
\limsup_{n\to\infty}\bigl(\|J_{r_{n+1}}y_{n+1} - J_{r_n}y_n\| - \|x_n - x_{n+1}\|\bigr) \le 0.
$$

It follows from Lemma 2.1 that

$$
\lim_{n\to\infty}\|J_{r_n}y_n - x_n\| = 0. \tag{3.3}
$$

Notice that $\|y_n - x_n\| \le \alpha_n\|f(x_n)-x_n\| + \|e_{n+1}\|$. It follows from the restrictions (a) and (c) that

$$
\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.4}
$$

In view of $\|J_{r_n}y_n - y_n\| \leq \|J_{r_n}y_n - x_n\| + \|x_n - y_n\|$, we find from (3.3) and (3.4) that

$$\lim_{n\to\infty} \|J_{r_n}y_n - y_n\| = 0. \tag{3.5}$$

Take a fixed number s such that $r > s > 0$. It follows from Lemma 2.3 that

$$\begin{aligned}\|y_n - J_s y_n\| &\leq \|y_n - J_{r_n}y_n\| + \left\| J_s\left(\frac{s}{r_n}y_n + \left(1 - \frac{s}{r_n}\right)J_{r_n}y_n\right) - J_s y_n\right\| \\ &\leq \|y_n - J_{r_n}y_n\| + \left\|\left(1 - \frac{s}{r_n}\right)(J_{r_n}y_n - y_n)\right\| \\ &\leq 2\|y_n - J_{r_n}y_n\|.\end{aligned}$$

This implies from (3.5) that

$$\lim_{n\to\infty} \|y_n - J_s y_n\| = 0. \tag{3.6}$$

Now, we are in a position to claim that $\limsup_{n\to\infty}\langle \bar{x} - f(\bar{x}), j(y_n - \bar{x})\rangle \leq 0$, where $\bar{x} = \lim_{t\to 0} x_t$, and x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)J_s x_t$, $\forall t \in (0,1)$. It follows that

$$\begin{aligned}\|x_t - y_n\|^2 &\leq (1-t)\bigl(\|x_t - y_n\|^2 + \|J_s y_n - y_n\|\|x_t - y_n\|\bigr) \\ &\quad + t\bigl\langle f(x_t) - x_t, j(x_t - y_n)\bigr\rangle + t\|x_t - y_n\|^2 \\ &\leq \|x_t - y_n\|^2 + \|J_s y_n - y_n\|\|x_t - y_n\| + t\bigl\langle f(x_t) - x_t, j(x_t - y_n)\bigr\rangle.\end{aligned}$$

This implies that $\langle x_t - f(x_t), j(x_t - y_n)\rangle \leq \frac{1}{t}\|J_s y_n - y_n\|\|x_t - y_n\|$, $\forall t \in (0,1)$. In view of (3.6), we find that

$$\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), j(x_t - y_n)\bigr\rangle \leq 0. \tag{3.7}$$

Since $x_t \to \bar{x}$, as $t \to 0$ and the fact that j is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}&\bigl|\bigl\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\bigr\rangle - \bigl\langle x_t - f(x_t), j(x_t - y_n)\bigr\rangle\bigr| \\ &\quad \leq \bigl|\bigl\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\bigr\rangle - \bigl\langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\bigr\rangle\bigr| \\ &\qquad + \bigl|\bigl\langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\bigr\rangle - \bigl\langle x_t - f(x_t), j(x_t - y_n)\bigr\rangle\bigr| \\ &\quad \leq \bigl|\bigl\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x}) - j(y_n - x_t)\bigr\rangle\bigr| + \bigl|\bigl\langle f(\bar{x}) - \bar{x} + x_t - f(x_t), j(y_n - x_t)\bigr\rangle\bigr| \\ &\quad \leq \bigl\|f(\bar{x}) - \bar{x}\bigr\|\bigl\|j(y_n - \bar{x}) - j(y_n - x_t)\bigr\| + \bigl\|f(\bar{x}) - \bar{x} + x_t - f(x_t)\bigr\|\|y_n - x_t\| \\ &\quad \to 0 \quad \text{as } t \to 0.\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists $\lambda > 0$ such that $\forall t \in (0,\lambda)$ the following inequality holds $\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \langle x_t - f(x_t), j(x_t - y_n)\rangle + \epsilon$. Taking $\limsup_{n\to\infty}$ in the above inequality, we find that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - y_n)\rangle + \epsilon$. Since ϵ is arbitrary, we obtain from (3.7) that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq 0$.

Finally, we prove that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\|y_n - \bar{x}\|^2 \le 2\alpha_n\big\langle f(x_n) - \bar{x}, j(y_n - \bar{x})\big\rangle + (1-\alpha_n)\|x_n - \bar{x}\|^2 + 2\|e_{n+1}\|\|y_n - \bar{x}\|. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \beta_n\big\langle x_n - \bar{x}, j(x_{n+1} - \bar{x})\big\rangle + (1-\beta_n)\big\langle J_{r_n}y_n - \bar{x}, j(x_{n+1} - \bar{x})\big\rangle \\ &\le \frac{\beta_n}{2}\big(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\big) + \frac{1-\beta_n}{2}\big(\|y_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\big).\end{aligned}$$

It follows from (3.8) that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \le{}& \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - \bar{x}\|^2 + 2\alpha_n(1-\beta_n)\big\langle f(x_n) - \bar{x}, j(y_n - \bar{x})\big\rangle \\ &+ 2\|e_{n+1}\|\|y_n - \bar{x}\|.\end{aligned}$$

In view of Lemma 2.4, we find the desired conclusion immediately. □

4 Applications

In this section, we give two applications of our main result in the framework of Hilbert spaces.

First, we consider, in the framework of Hilbert spaces, solutions of a Ky Fan inequality, which is known as an equilibrium problem in the terminology of Blum and Oettli; see [26] and [27] and the references therein.

Let C be a nonempty closed and convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

To study equilibrium problem (4.1), we may assume that F satisfies the following restrictions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

The following lemma can be found in [27].

Lemma 4.1 *Let C be a nonempty, closed, and convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $s > 0$ and $x \in H$, there exists $z \in C$ such that $F(z,y) + \frac{1}{s}\langle y - z, z - x\rangle \ge 0$, $\forall y \in C$. Further, define*

$$T_s x = \left\{z \in C : F(z,y) + \frac{1}{s}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\} \tag{4.2}$$

for all $s > 0$ and $x \in H$. Then (1) *T_s is single-valued and firmly nonexpansive*; (2) *$F(T_s) = EP(F)$ is closed and convex.*

Lemma 4.2 [28] *Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_F is a maximal monotone operator with domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$, where $EP(F)$ stands for the solution set of (4.1), *and*

$$T_s x = (I + sA_F)^{-1}x, \quad \forall x \in H, s > 0,$$

where T_s is defined as in (4.2).

Theorem 4.3 *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $f : C \to C$ be a fixed α-contraction. Let $\{x_n\}$ be a sequence generated in the following manner*: $x_0 \in C$ *and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{r_n}\big(\alpha_n f(x_n) + (1 - \alpha_n)x_n + e_{n+1}\big), \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$, *$\{e_n\}$ is a sequence in H, $\{r_n\}$ is a positive real number sequence, and $T_{r_n} = (I + r_n A_F)^{-1}$. Assume that $EP(F)$ is not empty and the above control sequences satisfy the restrictions* (a), (b), (c) *and* (d) *in Theorem* 3.1. *Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in EP(F)$, which is the unique solution to the following variational inequality $\langle f(\bar{x}) - \bar{x}, p - \bar{x}\rangle \leq 0$, $\forall p \in A^{-1}(0)$.*

Next, we consider the problem of finding a minimizer of a proper convex lower semicontinuous function.

For a proper lower semicontinuous convex function $g : H \to (-\infty, \infty]$, the subdifferential mapping ∂g of g is defined by

$$\partial g(x) = \big\{x^* \in H : g(x) + \big\langle y - x, x^*\big\rangle \leq g(y), \forall y \in H\big\}, \quad \forall x \in H.$$

Rockafellar [2] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ if and only if $g(v) = \min_{x \in H} g(x)$.

Theorem 4.4 *Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial g)^{-1}(0)$ is not empty. Let $f : H \to H$ be a κ-contraction, and let $\{x_n\}$ be a sequence in H in the following process*: $x_0 \in H$ *and*

$$\begin{cases} y_n = \arg\min_{z \in H}\{g(z) + \frac{\|z - \alpha_n f(x_n) - (1-\alpha_n)x_n - e_{n+1}\|^2}{2r_n}\}, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$, *$\{e_n\}$ is a sequence in E, and $\{r_n\}$ is a positive real number sequence. Assume that the above control sequences satisfy the restrictions in Theorem* 3.1. *Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in (\partial f)^{-1}(0)$, which is the unique solution to the following variational inequality $\langle f(\bar{x}) - \bar{x}, j(p - \bar{x})\rangle \leq 0$, $\forall p \in (\partial f)^{-1}(0)$.*

Proof Since $g: H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Note that

$$y_n = \arg\min_{z\in H}\left\{g(z) + \frac{\|z - \alpha_n f(x_n) - (1-\alpha_n)x_n - e_{n+1}\|^2}{2r_n}\right\}$$

is equivalent to

$$0 \in \partial g(y_n) + \frac{1}{r_n}\big(y_n - \alpha_n f(x_n) - (1-\alpha_n)x_n - e_{n+1}\big).$$

It follows that

$$\alpha_n f(x_n) + (1-\alpha_n)x_n + e_{n+1} \in y_n + r_n \partial g(y_n).$$

Following the proof in Theorem 3.1, we draw the desired conclusion immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this manuscript. Both authors read and approved the final manuscript.

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