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Strong and Δ-convergence for mixed type total asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract

It is our purpose in this paper first to introduce the class of *total asymptotically nonexpansive nonself mappings* and to prove the *demiclosed principle* for such mappings in CAT(0) spaces. Then, a new *mixed Agarwal-O'Regan-Sahu type iterative scheme* for approximating a common fixed point of two total asymptotically nonexpansive mappings and two total asymptotically nonexpansive nonself mappings is constructed. Under suitable conditions, some strong convergence theorems and Δ-convergence theorems are proved in a CAT(0) space. Our results improve and extend the corresponding results of Agarwal, O'Regan and Sahu (J. Nonlinear Convex Anal. 8(1):61-79, 2007), Guo *et al.* (Fixed Point Theory Appl. 2012:224, 2012. doi:10.1186/1687-1812-2012-224), Sahin *et al.* (Fixed Point Theory Appl. 2013:12, 2013. doi:10.1186/1687-1812-2013-12), Chang *et al.* (Appl. Math. Comput. 219:2611-2617, 2012), Khan and Abbas (Comput. Math. Appl. 61:109-116, 2011), Khan *et al.* (Nonlinear Anal. 74:783-791, 2011), Xu (Nonlinear Anal., Theory Methods Appl. 16(12):1139-1146, 1991), Chidume *et al.* (J. Math. Anal. Appl. 280:364-374, 2003) and others.

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1 Introduction and preliminaries

Let (X,d) be a metric space and $x,y\in X$ with $d(x,y)=l$. A *geodesic path* from x to y is an isometry $c:[0,l]\to X$ such that $c(0)=x$ and $c(l)=y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a (uniquely) *geodesic space* if every two points of X are joined by only one geodesic segment. A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1,x_2,x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\Delta(x_1,x_2,x_3)$ is the triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean space $\mathcal{R}^2$ such that

$$d(x_i,x_j)=d_{\mathcal{R}^2}(\bar{x_i},\bar{x_j}),\quad \forall i,j=1,2,3.$$

A geodesic space X is a CAT(0) *space* if for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in X and its comparison triangle $\bar{\Delta} := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathcal{R}^2$, the CAT(0) *inequality*

$$d(x,y) \leq d_{\mathcal{R}^2}(\bar{x}, \bar{y}) \tag{1.1}$$

is satisfied for all $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$.

The initials of the term 'CAT' are in honor of Cartan, Alexanderov and Toponogov. A CAT(0) space is a generalization of the Hadamard manifold, which is a simply connected, complete Riemannian manifold such that the sectional curvature is nonpositive. A thorough discussion of these spaces and their important role in various branches of mathematics are given in [1].

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y), d(z,y) = (1-t)d(x,y). \tag{1.2}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$.

A subset C of a CAT(0) space is convex if $[x,y] \subset C$ for all $x, y \in C$. For elementary facts about CAT(0) spaces, we refer the readers to [1] or [2].

The following lemma plays an important role in our paper.

Lemma 1.1 [2] *A geodesic space X is a* CAT(0) *space if and only if the following inequality holds*:

$$d^2\big((1-t)x \oplus ty, z\big) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{1.3}$$

for all $x, y, z \in X$ and all $t \in [0,1]$. In particular, if x, y, z are points in a CAT(0) *space and $t \in [0,1]$, then*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z). \tag{1.4}$$

Let (X,d) be a metric space, and let C be a nonempty subset of X. Recall that C is said to be a *retract of X* if there exists a continuous map $P : X \to C$ such that $Px = x$, $\forall x \in C$. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x,y), \quad \forall x, y \in C.$$

$T : C \to C$ is said to be *asymptotically nonexpansive* if there is a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d\big(T^n x, T^n y\big) \leq k_n d(x,y), \quad \forall n \geq 1, x, y \in C.$$

$T: C \to X$ is said to be an *asymptotically nonexpansive nonself mapping* if there is a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le k_n d(x,y), \quad \forall n \ge 1, x, y \in C,$$

where P is a nonexpansive retraction of X onto C.

$T: C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le L d(x,y), \quad \forall n \ge 1, x, y \in C. \tag{1.5}$$

Definition 1.2 A self-mapping $T: C \to C$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)$-*total asymptotically nonexpansive* if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta: [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n, \quad \forall n \ge 1, x, y \in C. \tag{1.6}$$

Definition 1.3 $T: C \to X$ is said to be a $(\{\mu_n\}, \{\nu_n\}, \zeta)$-*total asymptotically nonexpansive nonself mapping* if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta: [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n, \quad \forall n \ge 1, x, y \in C, \tag{1.7}$$

where P is a nonexpansive retraction of X onto C.

Definition 1.4 A nonself mapping $T: C \to X$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d\bigl(T(PT)^{n-1}x, T(PT)^{n-1}y\bigr) \le L d(x,y), \quad \forall n \ge 1, x, y \in C, \tag{1.8}$$

where P is a nonexpansive retraction of X onto C.

Remark 1.5 From the definitions, it is to know that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$, $n \ge 1$ and $\zeta(t) = t$, $t \ge 0$.

In 1976, Lim [3] introduced the concept of Δ-convergence in a general metric space. In 2008, Kirk and Panyanak [4] specialized Lim's concept to CAT(0) spaces and proved that it is very similar to the weak convergence in a Banach space setting.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [5, 6]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the existence problem of fixed point and the Δ-convergence problem of iterative sequences to a fixed point for nonexpansive mappings, asymptotically nonexpansive mappings in a CAT(0) space have been rapidly developed and many papers have appeared (see, *e.g.*, [7–26]).

The purpose of this paper is first to introduce the class of *total asymptotically nonexpansive nonself mappings* and to prove the *demiclosed principle* for such mappings in CAT(0) spaces. Then, a new *mixed Agarwal-O'Regan-Sahu type iterative scheme* [27] for approximating a common fixed point of two total asymptotically nonexpansive mappings and two total asymptotically nonexpansive nonself mappings is constructed. Under suitable conditions, some strong convergence theorems and Δ-convergence theorems are proved in a CAT(0) space. Our results extend and improve the corresponding results of Agarwal, O'Regan and Sahu [27], Guo *et al.* [28], Sahin [26], Chang *et al.* [24], Khan and Abbas [22], Khan *et al.* [23], Chidume *et al.* [29], Xu [30], Chang *et al.* [31] and many other recent results.

2 Demiclosed principle for total asymptotically nonexpansive nonself mappings

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}. \tag{2.1}$$

The *asymptotic radius* $r_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}. \tag{2.2}$$

The *asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.3}$$

And the *asymptotic center* $A_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}. \tag{2.4}$$

Proposition 2.1 [7] *Let X be a complete* CAT(0) *space, let* $\{x_n\}$ *be a bounded sequence in X and let C be a closed convex subset of X. Then*

(1) *there exists a unique point* $u \in C$ *such that*

$$r(u, \{x_n\}) = \inf_{x\in C} r(x, \{x_n\});$$

(2) $A(\{x_n\})$ *and* $A_C(\{x_n\})$ *both are singleton.*

Definition 2.2 [3, 4] Let X be a CAT(0) space. A sequence $\{x_n\}$ in X is said to Δ-*converge to* $p \in X$ if p is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = p$ and call p *the* Δ-*limit of* $\{x_n\}$.

Lemma 2.3

(1) *Let X be a complete* CAT(0) *space, let C be a closed convex subset of X. If* $\{x_n\}$ *is a bounded sequence in C, then the asymptotic center of* $\{x_n\}$ *is in C* [8];

(2) *Every bounded sequence in a complete* CAT(0) *space always has a* Δ*-convergent subsequence* [4].

Remark 2.4 Let X be a CAT(0) space and let C be a closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in C. In what follows, we denote it by

$$\{x_n\} \rightharpoonup w \quad \Leftrightarrow \quad \Phi(w) = \inf_{x \in C} \Phi(x), \tag{2.5}$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$.

Now we give a connection between the '$\rightharpoonup$' convergence and Δ-convergence.

Proposition 2.5 *Let X be a* CAT(0) *space, let C be a closed convex subset of X and let $\{x_n\}$ be a bounded sequence in C. Then $\Delta - \lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.*

Proof In fact, if $\Delta - \lim_{n\to\infty} x_n = p$, then it follows from Lemma 2.3 that $p \in C$. Since $A(\{x_n\}) = \{p\}$, we have $r(\{x_n\}) = r(p, \{x_n\})$. This implies that $\Phi(p) = \inf_{y\in C} \Phi(y)$, *i.e.*, $\{x_n\} \rightharpoonup p$. The desired conclusion is obtained. □

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's *demiclosed principle* [32] which states that if X is a uniformly convex Banach space, C is a nonempty closed convex subset of X, and $T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, *i.e.*, for any sequence $\{x_n\}$ in C if $x_n \to x$ weakly and $\|(I - T)x_n\| \to 0$, then $x = Tx$.

Later, Xu [30] and Chang *et al.* [31] proved the demiclosed principle for asymptotically nonexpansive mappings in a uniformly convex Banach space. In 2003, Chidume *et al.* [29] proved the demiclosed principle for asymptotically nonexpansive nonself mappings in uniformly convex Banach spaces.

In this section, by using the convergence '$\rightharpoonup$' defined by (2.5), we prove the *demiclosed principle* for total asymptotically nonexpansive nonself mappings in CAT(0) spaces, which extends the results of Xu [30], Chang *et al.* [31] and Chidume *et al.* [29] to CAT(0) spaces.

Theorem 2.6 (Demiclosed principle for total asymptotically nonexpansive nonself mappings in CAT(0) spaces) *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X, and let $T : C \to X$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive nonself mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\{x_n\} \rightharpoonup p$ defined by* (2.5) *and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tp = p$.*

Proof By the definition and Proposition 2.1, $\{x_n\} \rightharpoonup p$ if and only if $A_C(\{x_n\}) = \{p\}$. By Lemma 2.3, we have $A(\{x_n\}) = \{p\}$.

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n\to\infty} d\bigl(x_n, T(PT)^{m-1}x_n\bigr) = 0 \quad \text{for each } m \geq 1. \tag{2.6}$$

In fact, it is obvious that the conclusion is true for $m = 1$. Suppose the conclusion holds for $m \geq 1$, now we prove that the conclusion is also true for $m + 1$.

Indeed, since $x_n \in C$, we have $x_n = Px_n$. In addition, since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d\big(x_n, T(PT)^m x_n\big) &\le d\big(x_n, T(PT)^{m-1} x_n\big) + d\big(T(PT)^{m-1} x_n, T(PT)^m x_n\big) \\ &\le d\big(x_n, T(PT)^{m-1} x_n\big) + Ld(x_n, PTx_n) \\ &= d\big(x_n, T(PT)^{m-1} x_n\big) + Ld(Px_n, PTx_n) \\ &\le d\big(x_n, T(PT)^{m-1} x_n\big) + Ld(x_n, Tx_n) \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

Equation (2.6) is proved. Hence for each $x \in X$ and $m \ge 1$, we have

$$\Phi(x) := \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d\big(T(PT)^{m-1}(x_n), x\big). \tag{2.7}$$

In (2.7), taking $x = T(PT)^{m-1}p$, $m \ge 1$, we have

$$\begin{aligned} \Phi\big(T(PT)^{m-1}p\big) &= \limsup_{n\to\infty} d\big(T(PT)^{m-1} x_n, T(PT)^{m-1} p\big) \\ &\le \limsup_{n\to\infty} \big\{ d(x_n, p) + \nu_m \zeta\big(d(x_n, p)\big) + \mu_m \big\}. \end{aligned}$$

Letting $m \to \infty$ and taking superior limit on both sides, we get that

$$\limsup_{m\to\infty} \Phi\big(T(PT)^{m-1}p\big) \le \Phi(p). \tag{2.8}$$

Furthermore, for any $n, m \ge 1$, it follows from inequality (1.3) with $t = \frac{1}{2}$ that

$$\begin{aligned} & d^2\left(x_n, \frac{p \oplus T(PT)^{m-1}(p)}{2}\right) \\ & \quad \le \frac{1}{2} d^2(x_n, p) + \frac{1}{2} d^2\big(x_n, T(PT)^{m-1}(p)\big) - \frac{1}{4} d^2\big(p, T(PT)^{m-1}(p)\big). \end{aligned} \tag{2.9}$$

Letting $n \to \infty$ and taking superior limit on both sides of the above inequality, for any $m \ge 1$, we get

$$\begin{aligned} & \Phi\left(\frac{p \oplus T(PT)^{m-1}(p)}{2}\right)^2 \\ & \quad \le \frac{1}{2}\Phi(p)^2 + \frac{1}{2}\Phi\big(T(PT)^{m-1}(p)\big)^2 - \frac{1}{4} d^2\big(p, T(PT)^{m-1}(p)\big). \end{aligned} \tag{2.10}$$

Since $A(\{x_n\}) = \{p\}$, for any $m \ge 1$, we have

$$\begin{aligned} \Phi(p)^2 &\le \Phi\left(\frac{p \oplus T(PT)^{m-1}(p)}{2}\right)^2 \\ &\le \frac{1}{2}\Phi(p)^2 + \frac{1}{2}\Phi\big(T(PT)^{m-1}(p)\big)^2 - \frac{1}{4} d^2\big(p, T(PT)^{m-1}(p)\big). \end{aligned} \tag{2.11}$$

This implies that

$$d^2\big(p, T(PT)^{m-1}(p)\big) \le 2\Phi\big(T(PT)^{m-1}(p)\big)^2 - 2\Phi(p)^2. \tag{2.12}$$

From (2.8) and (2.12), we have $\lim_{m\to\infty} d(p, T(PT)^{m-1}p) = 0$. Hence we have

$$\begin{aligned} d(Tp,p) &\le d\big(Tp, T(PT)^m p\big) + d\big(T(PT)^m p, p\big) \\ &\le Ld\big(p,(PT)^m p\big) + d\big(T(PT)^m p, p\big) \\ &= Ld\big(Pp,(PT)(PT)^{m-1} p\big) + d\big(T(PT)^m p, p\big) \\ &\le Ld\big(p, T(PT)^{m-1} p\big) + d\big(T(PT)^m p, p\big) \to 0 \quad (\text{as } m\to\infty), \end{aligned}$$

i.e., $p = Tp$ as desired. □

The following theorem can be obtained from Theorem 2.6 immediately which is a generalization of Kirk *et al.* [4, Proposition 3.7], Xu [30], Chang *et al.* [31] and Chidume *et al.* [29, Theorem 3.4].

Theorem 2.7 *Let C be a closed and convex subset of a complete* CAT(0) *space X. Let T be a mapping satisfying one of the following conditions*:

(1) *$T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$;*

(2) *$T : C \to X$ is an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$;*

(3) *$T : C \to C$ is a $(\{\nu_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive mapping.*

Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $\Delta-\lim_{n\to\infty} x_n = p$. *Then $Tp = p$.*

3 Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces

In this section we prove some Δ-convergence theorems for the *mixed Agarwal-O'Regan-Sahu type iterative scheme* [27]

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1} y_n), \qquad n \ge 1, \\ y_n = P((1-\beta_n)S_2^n x_n \oplus \beta_n T_2(PT_2)^{n-1} x_n), \end{cases} \tag{3.1}$$

where C is a nonempty bounded closed and convex subset of a complete CAT(0) space X, P is a nonexpansive retraction of X onto C, $T_i : C \to X$, $i = 1,2$, is a uniformly L_i-Lipschitzian and $(\{\nu_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive nonself mapping (defined by (1.7)), and $S_i : C \to C$, $i = 1,2$, is a uniformly $\tilde{L}_i$-Lipschitzian and $(\{\tilde{\nu}_n^{(i)}\},\{\tilde{\mu}_n^{(i)}\},\tilde{\zeta}^{(i)})$ total asymptotically nonexpansive mapping (defined by (1.6)) such that the following conditions are satisfied:

(1) $\sum_{n=1}^\infty \nu_n^{(i)} < \infty$, $\sum_{n=1}^\infty \mu_n^{(i)} < \infty$, $\sum_{n=1}^\infty \tilde{\nu}_n^{(i)} < \infty$, $\sum_{n=1}^\infty \tilde{\mu}_n^{(i)} < \infty$, $i = 1,2$;

(2) There exists a constant $M^* > 0$ such that $\zeta^{(i)}(r) \le M^* r$, $\tilde{\zeta}^{(i)}(r) \le M^* r$, $\forall r \ge 0$, $i = 1,2$.

Remark 3.1 Without loss of generality, in the sequel, we can assume that $S_i : C \to C$ and $T_i : C \to X$, $i = 1,2$, both are uniformly L-Lipschitzian and $(\{\nu_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive mappings satisfying the conditions (1) and (2). In fact, letting $\nu_n = \max\{\nu_n^{(i)}, \tilde{\nu}_n^{(i)}, i = 1,2\}$, $\mu_n = \max\{\mu_n^{(i)}, \tilde{\mu}_n^{(i)}, i = 1,2\}$, $L = \max\{L_i, \tilde{L}_i, i = 1,2\}$ and $\zeta =$

$\max\{\zeta^{(i)}, \tilde{\zeta}^{(i)}, i = 1, 2\}$, then $S_i : C \to C$ and $T_i : C \to X$, $i = 1, 2$, are the mappings satisfying the required conditions.

The following lemmas will be used to prove our main results.

Lemma 3.2 (Chang *et al.* [24]) *Let X be a* CAT(0) *space, $x \in X$ be a given point and $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1 - c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \qquad \limsup_{n\to\infty} d(y_n, x) \leq r \quad \text{and}$$
$$\lim_{n\to\infty} d\big((1 - t_n)x_n \oplus t_n y_n, x\big) = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.2}$$

Lemma 3.3 *Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + \lambda_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.4 [2] *Let X be a complete* CAT(0) *space, $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{p\}$, and $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

Now we are in a position to give the main results of this paper.

Theorem 3.5 *Let C be a bounded closed and convex subset of a complete* CAT(0) *X. Let $T_i : C \to X$, $i = 1, 2$, be a uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive nonself mapping, and let $S_i : C \to C$, $i = 1, 2$, be a uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mapping. If $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \cap F(S_i) \neq \emptyset$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;
(ii) *there exist constants $a, b \in (0, 1)$ with $0 < b(1 - c) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$;*
(iii) *there exists a constant $M^* > 0$ such that $\zeta(r) \leq M^* r$, $r \geq 0$;*
(iv) *$d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in C$ and $i = 1, 2$,*

then the sequence $\{x_n\}$ defined by (3.1) *Δ-converges to some point $p^* \in \mathcal{F}$ (a common fixed point of T_i and S_i, $i = 1, 2$).*

Proof (I) First we prove that the following limits exist

$$\lim_{n\to\infty} d(x_n, p) \quad \text{for each } p \in \mathcal{F} \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{F}). \tag{3.3}$$

In fact, since $p \in \mathcal{F}$, $p = Pp$. In addition, since S_i and T_i, $i = 1, 2$, are total asymptotically nonexpansive mappings, by the condition (iii), we have

$$\begin{aligned}
d(y_n, p) &= d\big(P\big((1-\beta_n)S_2^n x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n\big), Pp\big) \\
&\le d\big((1-\beta_n)S_2^n x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n, p\big) \\
&\le (1-\beta_n)d\big(S_2^n x_n, p\big) + \beta_n d\big(T_2(PT_2)^{n-1}x_n, p\big) \\
&= (1-\beta_n)\big\{d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n\big\} + \beta_n\big\{d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n\big\} \\
&= d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n \\
&\le \big(1 + \nu_n M^*\big)d(x_n,p) + \mu_n
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
d(x_{n+1}, p) &= d\big(P\big((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n\big), Pp\big) \\
&\le d\big((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n, p\big) \\
&\le (1-\alpha_n)d\big(S_1^n x_n, p\big) + \alpha_n d\big(T_1(PT_1)^{n-1}y_n, p\big) \\
&= (1-\alpha_n)\big\{d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n\big\} + \alpha_n\big\{d(y_n,p) + \nu_n\zeta\big(d(y_n,p)\big) + \mu_n\big\} \\
&\le (1-\alpha_n)\big\{\big(1 + \nu_n M^*\big)d(x_n,p) + \mu_n\big\} + \alpha_n\big\{\big(1 + \nu_n M^*\big)d(y_n,p) + \mu_n\big\}.
\end{aligned} \tag{3.5}$$

Substituting (3.4) into (3.5) and simplifying it, we have

$$d(x_{n+1}, p) \le (1 + \sigma_n)d(x_n, p) + \xi_n, \quad \forall n \ge 1 \text{ and } p \in \mathcal{F}, \tag{3.6}$$

and so

$$d(x_{n+1}, \mathcal{F}) \le (1 + \sigma_n)d(x_n, \mathcal{F}) + \xi_n, \quad \forall n \ge 1, \tag{3.7}$$

where $\sigma_n = \nu_n M^*(1 + \alpha_n(1 + \nu_n M^*))$, $\xi_n = (1 + \alpha_n(1 + \nu_n M^*))\mu_n$. By virtue of the condition (i),

$$\sum_{n=1}^{\infty} \sigma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \xi_n < \infty. \tag{3.8}$$

By Lemma 3.3 the limits $\lim_{n\to\infty} d(x_n, \mathcal{F})$ and $\lim_{n\to\infty} d(x_n, p)$ exist for each $p \in \mathcal{F}$.

(II) Next we prove that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2. \tag{3.9}$$

In fact, it follows from (3.3) that for each given $p \in \mathcal{F}$, $\lim_{n\to\infty} d(x_n, p)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, p) = r \ge 0. \tag{3.10}$$

From (3.4) we have

$$\liminf_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le \lim_{n\to\infty} \big\{\big(1 + \nu_n M^*\big)d(x_n, p) + \mu_n\big\} = r. \tag{3.11}$$

Since

$$d\big(T_1(PT_1)^{n-1}y_n,p\big) = d\big(T_1(PT_1)^{n-1}y_n, T_1(PT_1)^{n-1}p\big) \le d(y_n,p) + \nu_n\zeta\big(d(y_n,p)\big) + \mu_n \\ \le \big(1+\nu_n M^*\big)d(y_n,p) + \mu_n, \quad \forall n \ge 1,$$

and

$$d\big(S_1^n x_n,p\big) \le d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n \le \big(1+\nu_n M^*\big)d(x_n,p) + \mu_n, \quad \forall n \ge 1,$$

then we have

$$\limsup_{n\to\infty} d\big(T_1(PT_1)^{n-1}y_n,p\big) \le r \tag{3.12}$$

and

$$\limsup_{n\to\infty} d\big(S_1^n x_n,p\big) \le r. \tag{3.13}$$

In addition, it follows from (3.6) that

$$d(x_{n+1},p) \le d\big((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n,p\big) \le (1+\sigma_n)d(x_n,p) + \xi_n.$$

This implies that

$$\lim_{n\to\infty} d\big((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n,p\big) = r. \tag{3.14}$$

From (3.12)-(3.14) and Lemma 3.2, one gets that

$$\lim_{n\to\infty} d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big) = 0. \tag{3.15}$$

By the same method, we can also prove that

$$\lim_{n\to\infty} d\big(S_2^n x_n, T_2(PT_2)^{n-1}x_n\big) = 0. \tag{3.16}$$

By virtue of the condition (iv), it follows from (3.15) and (3.16) that

$$\lim_{n\to\infty} d\big(x_n, T_1(PT_1)^{n-1}y_n\big) \le \lim_{n\to\infty} d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big) = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} d\big(x_n, T_2(PT_2)^{n-1}x_n\big) \le \lim_{n\to\infty} d\big(S_2^n x_n, T_2(PT_2)^{n-1}x_n\big) = 0. \tag{3.18}$$

Since $S_2^n x_n \in C$, $S_2^n x_n = PS_2^n x_n$. By (3.1) and (3.16) we have

$$d\big(y_n, S_2^n x_n\big) \le d\big((1-\beta_n)S_2^n x_n \oplus \beta_n T_2(PT_2)^{n-1}x_n, S_2^n x_n\big) \\ \le \beta_n d\big(T_2(PT_2)^{n-1}x_n, S_2^n x_n\big) \to 0 \quad (\text{as } n\to\infty). \tag{3.19}$$

Observe that

$$d(x_n, y_n) \le d\big(x_n, T_2(PT_2)^{n-1}x_n\big) + d\big(T_2(PT_2)^{n-1}x_n, S_2^n x_n\big) + d\big(S_2^n x_n, y_n\big).$$

From (3.18) and (3.19) we get

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.20}$$

This together with (3.17) implies that

$$\begin{aligned} d\big(x_n, T_1(PT_1)^{n-1}x_n\big) &\le d\big(x_n, T_1(PT_1)^{n-1}y_n\big) + d\big(T_1(PT_1)^{n-1}y_n, T_1(PT_1)^{n-1}x_n\big) \\ &= d\big(x_n, T_1(PT_1)^{n-1}y_n\big) + d(x_n, y_n) + \nu_n \zeta\big(d(x_n, y_n)\big) + \mu_n \\ &\le d\big(x_n, T_1(PT_1)^{n-1}y_n\big) + \big(1 + \nu_n M^*\big) d(x_n, y_n) + \mu_n \to 0. \end{aligned} \tag{3.21}$$

On the other hand, by the condition (iv), $d(x_n, T_1(PT_1)^{n-1}x_n) \le d(S_1^n x_n, T_1(PT_1)^{n-1}x_n)$. Hence from (3.17) and (3.20), we have

$$\begin{aligned} &d\big(S_1^n x_n, T_1(PT_1)^{n-1}x_n\big) \\ &\quad\le d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big) + d\big(T_1(PT_1)^{n-1}y_n, T_1(PT_1)^{n-1}x_n\big) \\ &\quad\le d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big) + Ld(y_n, x_n) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.22}$$

By the condition (iv), $d(x_n, T_1(PT_1)^{n-1}x_n) \le d(S_1^n x_n, T_1(PT_1)^{n-1}x_n)$. Hence from (3.22) we have that

$$d\big(S_1^n x_n, x_n\big) \le d\big(S_1^n x_n, T_1(PT_1)^{n-1}x_n\big) + d\big(T_1(PT_1)^{n-1}x_n, x_n\big) \to 0 \quad (\text{as } n \to \infty).$$

This together with (3.17) shows that

$$\begin{aligned} d(x_{n+1}, x_n) &\le d\big((1-\alpha_n)S_1^n x_n \oplus \alpha_n T_1(PT_1)^{n-1}y_n, x_n\big) \\ &\le (1-\alpha_n) d\big(S_1^n x_n, x_n\big) + \alpha_n d\big(T_1(PT_1)^{n-1}y_n, x_n\big) \to 0 \\ &\quad (\text{as } n \to \infty). \end{aligned} \tag{3.23}$$

Hence from (3.18), (3.21) and (3.23), for each $i = 1, 2$, we have

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i(PT_i)^n x_{n+1}\big) \\ &\quad + d\big(T_i(PT_i)^n x_{n+1}, T_i(PT_i)^n x_n\big) + d\big(T_i(PT_i)^n x_n, T_i x_n\big) \\ &\le (1+L) d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i(PT_i)^n x_{n+1}\big) + Ld\big((PT_i)^n x_n, x_n\big) \\ &= (1+L) d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i(PT_i)^n x_{n+1}\big) + Ld\big(PT_i(PT_i)^{n-1}x_n, Px_n\big) \\ &\le (1+L) d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i(PT_i)^n x_{n+1}\big) \\ &\quad + Ld\big(T_i(PT_i)^{n-1}x_n, x_n\big) \to 0. \end{aligned} \tag{3.24}$$

By virtue of the condition (iv), $d(S_i x_n, T_i(PT_i)^{n-1}x_n) \le d(S_i^n x_n, T_i(PT_i)^{n-1}x_n)$. It follows from (3.18), (3.21) and (3.22) that

$$\begin{aligned} d(x_n, S_i x_n) &\le d\bigl(x_n, T_i(PT_i)^{n-1}x_n\bigr) + d\bigl(S_i x_n, T_i(PT_i)^{n-1}x_n\bigr) \\ &\le d\bigl(x_n, T_i(PT_i)^{n-1}x_n\bigr) + d\bigl(S_i^n x_n, T_i(PT_i)^{n-1}x_n\bigr) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.25}$$

Equation (3.9) is proved.

(III) Now we prove that

$$\omega_w(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A\bigl(\{u_n\}\bigr) \subset \mathcal{F} \tag{3.26}$$

and $\omega_w(x_n)$ consists of exactly one point.

In fact, let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in C$. In view of (3.9), $\lim_{n\to\infty} d(v_n, T_i v_n) = 0$, $\lim_{n\to\infty} d(v_n, S_i v_n) = 0$, $i = 1, 2$. It follows from Theorem 2.7 that $v \in \mathcal{F}$. So, by (3.3), the limit $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 3.4 $u = v$. This implies that $\omega_w(x_n) \subset \mathcal{F}$.

Next we prove that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset \mathcal{F}$, from (3.3) the limit $\lim_{n\to\infty} d(x_n, u)$ exists. In view of Lemma 3.4, $x = u$. The conclusion is proved.

(IV) Finally we prove $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$.

In fact, it follows from (3.3) that $\{d(x_n, p)\}$ is convergent for each $p \in \mathcal{F}$. By (3.9) and (3.26), $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $\omega_w(x_n) \subset \mathcal{F}$ and $\omega_w(x_n)$ consists of exactly one point. This shows that $\{x_n\}$ Δ-converges to a point of $\mathcal{F}$.

The conclusion of Theorem 3.5 is proved. □

Remark 3.6 (1) Now we give an example which satisfies the condition (iv) in Theorem 3.5.

Let $C = [-1, 1]$ be a subset in $\mathcal{R}$. Define two mappings $S_1 = S_2 = S$, $T_1 = T_2 = T : C \to C$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0), \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1, 0). \end{cases}$$

It is proved in Guo [28] that both S and T are asymptotically nonexpansive mappings (therefore they are total asymptotically nonexpansive mappings) with $F(T) \cap F(S) \neq \emptyset$ and satisfy the condition (iv).

(2) Theorem 3.5 contains the main results of Sahin [26], Khan Abbas [22], Khan *et al.* [23] and Chang *et al.* [24] as its special cases. Theorem 3.5 also extends the main result of Guo *et al.* [28] from a Banach space to a CAT(0) space.

The following results can be obtained from Theorem 3.5 immediately.

Theorem 3.7 *Let C, X and $T_i : C \to X$, $i = 1, 2$ be the same as in Theorem* 3.5. *If $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) *there exist constants $a, b \in (0, 1)$ with $0 < b(1 - c) \le \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$.*

(iii) *there exists a constant $M^* > 0$ such that $\zeta(r) \le M^* r$, $r \ge 0$;*

then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1 - \alpha_n)x_n \oplus \alpha_n T_1(PT_1)^{n-1} y_n), \quad n \ge 1, \\ y_n = P((1 - \beta_n)x_n \oplus \beta_n T_2(PT_2)^{n-1} x_n), \end{cases} \tag{3.27}$$

Δ-converges to a common fixed point of T_1 and T_2.

Proof Take $S_i = I$ (the identity mapping on C) in Theorem 3.5 and note that in this case the condition (iv) in Theorem 3.5 is satisfied automatically. Hence the conclusion of Theorem 3.7 can be obtained from Theorem 3.5 immediately. □

Theorem 3.8 *Let C and X be the same as in Theorem* 3.5. *Let $T_i : C \to C$ and $S_i : C \to C$, $i = 1, 2$, be uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mappings. If $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \cap F(S_i) \neq \emptyset$ and the* (i)-(iv) *in Theorem* 3.5 *are satisfied, then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)S_1^n x_n \oplus \alpha_n T_1^n y_n, \quad n \ge 1, \\ y_n = (1 - \beta_n)S_2^n x_n \oplus \beta_n T_2^n x_n, \end{cases} \tag{3.28}$$

Δ-converges to a common fixed point of T_i and S_i, $i = 1, 2$.

Proof Since T_i, $i = 1, 2$, is a self-mapping from C to C, take $P = I$ (the identity mapping on C), then $T_i(PT_i)^{n-1} = T_i^n$. The conclusion of Theorem 3.8 is obtained from Theorem 3.5. □

Remark 3.9 Theorem 3.8 improves and extends the main results of Agawal O'Regan Sahu [27] from a Banach space to a CAT(0) space. As well as it also extends and improves the main results in Sahin [26].

4 Strong convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces

Recall that a mapping $T : C \to X$ is said to be *demi-compact* if for any sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly (*i.e.*, in metric topology) to some point $x^* \in C$.

Theorem 4.1 *Under the assumptions of Theorem* 3.5, *if one of S_1, S_2, T_1 and T_2 is demi-compact, then the sequence defined by* (3.1) *converges strongly* (*i.e., in metric topology*) *to a common fixed point $p \in \mathcal{F}$.*

Proof By virtue of (3.9): $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$, $i = 1,2$ and one of S_1, S_2, T_1 and T_2 is demi-compact, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $p \in C$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, for each $i = 1, 2$, we have

$$d(p, S_i p) = \lim_{n\to\infty} d(x_{n_i}, S_i x_{n_i}) = 0,$$

$$d(p, T_i p) = \lim_{n\to\infty} d(x_{n_i}, T_i x_{n_i}) = 0.$$

This implies that $p \in \mathcal{F}$. Again by (3.3) the limit $\lim_{n\to\infty} d(x_n, p)$ exists. Hence we have $\lim_{n\to\infty} d(x_n, p) = 0$. This completes the proof of Theorem 4.1. □

Theorem 4.2 *Under the assumptions of Theorem* 3.5, *if there exists a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0, f(r) > 0, \forall r > 0$ *such that*

$$f\big(d(x,\mathcal{F})\big) \le d(x, S_1 x) + d(x, S_2 x) + d(x, T_1 x) + d(x, T_2 x), \quad \forall x \in C, \tag{4.1}$$

then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly* (*i.e., in metric topology*) *to a common fixed point* $p^* \in \mathcal{F}$.

Proof It follows from (3.9) that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2.$$

Therefore we have $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$, $r > 0$, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, it follows from (3.6) that for any $p \in \mathcal{F}$

$$d(x_{n+1}, p) \le (1 + \sigma_n) d(x_n, p) + \xi_n, \quad \forall n \ge 1,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Hence for any positive integers n, m, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, p) + d(x_n, p) \\ &\le (1 + \sigma_{n+m-1}) d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p). \end{aligned}$$

Since for each $x \ge 0$, $1 + x \le e^x$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le e^{\sigma_{n+m-1}} d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p) \\ &\le e^{\sigma_{n+m-1} + \sigma_{n+m-2}} d(x_{n+m-2}, p) + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\ &\le \cdots \\ &\le e^{\sum_{i=n}^{n+m-1} \sigma_i} d(x_n, p) + e^{\sum_{i=n+1}^{n+m-1} \sigma_i} \xi_n + e^{\sum_{i=n+2}^{n+m-2} \sigma_i} \xi_{n+1} + \cdots \\ &\quad + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\ &\le (1 + M) d(x_n, p) + M \sum_{i=n}^{n+m-1} \xi_i, \end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}\sigma_i} < \infty$. By (3.3) $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Therefore we have

$$d(x_{n+m}, x_n) \leq (1+M)d(x_n, \mathcal{F}) + M\sum_{i=n}^{n+m-1}\xi_i \to 0 \quad (\text{as } n, m \to \infty).$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in a complete CAT(0) space X, it is complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly (*i.e.*, in metric topology in X) to some point $p^* \in C$. It is easy to prove that $F(T_i)$ and $F(S_i)$, $i = 1, 2$ are closed subsets in C, so is $\mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, $p^* \in \mathcal{F}$. This completes the proof of Theorem 4.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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