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New iterative schemes for a finite family of nonself uniformly quasi-Lipschitzian mappings in Banach spaces

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Abstract

In this article, we introduce the concept of nonself uniformly quasi-Lipschitzian mapping and consider a new iterative scheme with errors to converge to a common fixed point for a finite family of nonself uniformly quasi-Lipschitzian mappings in Banach spaces. The results of this article improve and extend many known results.

Keywords: nonself uniformly quasi-Lipschitzian mapping, new iterative scheme with errors, common fixed point, Banach spaces

1 Introduction and preliminaries

Throughout the article, we assume that X is a real Banach space, C is a nonempty subset of X, and Fix(T) is the set of fixed points of mapping T, i.e., $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Definition 1.1. *Let $T : C \to C$ be a mapping.*

(1) *T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T^n x - T^n y|| \le k_n ||x - y||$$

for all $x, y \in C$ and $n \ge 1$.

(2) *T is said to be uniformly Lipschitzian if there exists a constant $L > 0$ such that*

$$||T^n x - T^n y|| \le L||x - y||$$

for all $x, y \in C$ and $n \ge 1$.

(3) *T is called asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T^n x - p|| \le k_n ||x - p||$$

for all $x \in C$, $p \in \mathrm{Fix}(T)$ and $n \ge 1$.

Remark 1.1. (i) The concept of asymptotically nonexpansive mapping was initially introduced by Geobel and Kirk [1]. Meanwhile, they proved that if C is a nonempty closed, convex, and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping has a fixed point.

(ii) It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly Lipschitzian mapping (taking $L = \sup_{n \geq 1} k_n$), and if $\mathrm{Fix}(T) \neq \emptyset$, then every asymptotically nonexpansive mapping T is an asymptotically quasi-nonexpansive mapping.

Definition 1.2. *Let X be a real Banach space and C be a nonempty subset of X.*

(1) *A mapping $P : X \to C$ is said to be retraction if $P^2 = P$.*
(2) *If there exists a nonexpansive retraction $P : X \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be nonexpansive retract of X.*
Next, we introduce some concepts for nonself mappings.

Definition 1.3. *Let X be a real Banach space, C be a nonempty subset of X, and $P : X \to C$ the nonexpansive retraction of X onto C. Let $T : C \to X$ be a nonself mapping.*

(1) *T is said to be nonself asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq k_n||x - y||$$

for all $x, y \in C$ and $n \geq 1$.
(2) *T is said to be nonself uniformly Lipschitzian if there exists a constant $L > 0$ such that*

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq L||x - y||$$

for all $x, y \in C$ and $n \geq 1$.
(3) *T is said to be nonself asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T(PT)^{n-1}x - p|| \leq k_n||x - p||$$

for all $x \in C$, $p \in \mathrm{Fix}(T)$ and $n \geq 1$.
(4) *T is said to be nonself uniformly quasi-Lipschitzian if there exists a constant $L > 0$ such that*

$$||T(PT)^{n-1}x - p|| \leq L||x - p||$$

for all $x \in C$, $p \in \mathrm{Fix}(T)$ and $n \geq 1$.

Remark 1.2. (i) The concept of nonself asymptotically nonexpansive mapping was introduced by Chidume et al. [2] which is a generalization of asymptotically nonexpansive self-mapping.

(ii) If T is a nonself asymptotically nonexpansive mapping, then it must be nonself uni-formly Lipschitzian, but the converse does not hold [3].
(iii) If T is a nonself uniformly Lipschitzian mapping or a nonself asymptotically quasi-nonexpansive mapping, then it must be a nonself uniformly quasi-Lipschitzian mapping.

Fixed points iterative technique for (self or nonself) asymptotically nonexpansive mappings in Banach spaces, including Mann type iteration, Ishikawa type iteration, and three-step type iteration, have been studied by many authors (see, e.g., [2-8]). Recently, Khan et al. [9] introduced an iterative scheme (which generalizes Mann iteration, Ishikawa iteration, and three-step iteration) for a finite family of asymptotically quasi-nonexpansive self-mappings $\{T_i : i \in I\}$: $C \to C$, where $I = \{1, 2, \ldots, k\}$ and C be a convex set. For any initial point $x_1 \in C$:

$$\begin{cases} x_{n+1} = (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} = (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{(k-1)}^n y_{(k-2)n}, \\ y_{(k-2)n} = (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{(k-2)}^n y_{(k-3)n}, \\ \quad \vdots \\ y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{cases} \tag{1.1}$$

where $y_{0n} = x_n$ and $\{\alpha_{in}\}$ are real sequences in [0, 1] for all $n \geq 1$. They proved the convergence to a common fixed point for a finite family of asymptotically quasi-nonexpansive self-mappings in Banach spaces by using the iterative (1.1).

Inspired and motivated by the above research, we introduce a new iterative process as follows:

Let C be a nonempty convex subset of a real Banach space X and $P : X \to C$ the nonexpansive retraction of X onto C. Assume $T_i : C \to X$, $i \in I$ be a finite family of nonself uniformly quasi-Lipschitzian mappings. For any $x_1 \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = y_{kn} = P(\alpha_{kn}x_n + \beta_{kn}T_k(PT_k)^{n-1}y_{(k-1)n} + \gamma_{kn}u_{kn}), \\ y_{(k-1)n} = P(\alpha_{(k-1)n}x_n + \beta_{(k-1)n}T_{(k-1)}(PT_{(k-1)})^{n-1}y_{(k-2)n} + \gamma_{(k-1)n}u_{(k-1)n}), \\ y_{(k-2)n} = P(\alpha_{(k-2)n}x_n + \beta_{(k-2)n}T_{(k-2)}(PT_{(k-2)})^{n-1}y_{(k-3)n} + \gamma_{(k-2)n}u_{(k-2)n}), \\ \vdots \\ y_{1n} = P(\alpha_{1n}x_n + \beta_{1n}T_1(PT_1)^{n-1}y_{0n} + \gamma_{1n}u_{1n}), \end{cases} \tag{1.2}$$

where $y_{0n} = x_n$ for all $n \geq 1$, $\{\alpha_{in}\}$, $\{\beta_{in}\}$, $\{\gamma_{in}\}$ are real sequences in [0, 1] with $\alpha_{in} + \beta_{in} + \gamma_{in} = 1$ and $\{u_{in}\}$ is a bounded sequence in C, for $i \in I$.

Remark 1.3. The iterative sequence (1.2) is a natural generalization of the well-known iteration:

(i) If $\{T_i : i \in I\}$ is asymptotically quasi-nonexpansive self-mappings and $\gamma_{in} = 0$ for $i \in I$ and $n \geq 1$, then the iterative sequence (1.2) reduces to (1.1).
(ii) If $k = 2$ and T_1, T_2 are nonself asymptotically nonexpansive mappings, $\gamma_{1n} = \gamma_{2n} = 0$, then the iterative sequence (1.2) reduces to the Ishikawa type iteration in Wang [8].
(iii) If $k = 1$ and T_1 is a nonself asymptotically nonexpansive mapping, $\gamma_{1n} = 0$, then the iterative sequence (1.2) reduce to the Mann type iteration in Chidume et al. [2].

In this article, we are concerned with the convergence to a common fixed point for a finite family of nonself uniformly quasi-Lipschitzian mappings in Banach spaces by using the iterative sequence (1.2). As one will see, our results extend and generalize the corresponding results in [2-10] as follows: (i) the condition $\sum_{n=1}^{\infty}(k_{in}-1)<\infty$ is dropped; (ii) the condition $\sum_{n=1}^{\infty}\gamma_{in}<\infty, i\in I$ is replaced with $\sum_{n=1}^{\infty}(\beta_{kn}+\gamma_{kn})<\infty$; (iii) a more general mapping is considered.

We need the following lemma for proving our main results.

Lemma 1.1. ([5]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the following conditions:*

$$a_{n+1}\le(1+b_n)a_n+c_n, \forall n\ge 1.$$

where $\sum_{n=1}^{\infty}b_n<\infty$ and $\sum_{n=1}^{\infty}c_n<\infty$, then $\lim_{n\to\infty}a_n$ exists.

2 Main results

In this section, we shall prove the strong convergence of the iteration sequence (1.2) to a common fixed point for a finite family of nonself uniformly quasi-Lipschitzian mappings T_i $(i\in I)$ in real Banach spaces. We first prove the following lemma.

Lemma 2.1. *Let C be a nonempty convex subset of a real normed linear space X. Let $\{T_i : i\in I\}$: $C\to X$ be a finite family of nonself uniformly quasi-Lipschitzian mappings with $L_i>0$, i.e.,*

$$\| T_i(PT_i)^{n-1}x-p_i\|\le L_i\| x-p_i\|$$

for all $x\in C$ and $p_i\in Fix(T_i)$, $i\in I$. Define the sequence $\{x_n\}$ as in (1.2) with $\sum_{n=1}^{\infty}(\beta_{kn}+\gamma_{kn})<\infty$. If $F=\bigcap_{i=1}^{k}Fix(T_i)\neq\emptyset$, then

(i) *there exist two constants M_0, $M_1>0$ such that*

$$||x_{n+1}-p||\le[1+\theta_nM_0]||x_n-p||+\theta_nM_1 \tag{2.1}$$

where $\theta_n=\beta_{kn}+\gamma_{kn}$ for all $n\ge 1$, $p\in F$.

(ii) *there exists a constant $M_2>0$, such that*

$$||x_{n+m}-p||\le M_2||x_n-p||+M_1M_2\sum_{j=n}^{n+m-1}\theta_j \tag{2.2}$$

for all n, $m\ge 1$, $p\in F$.

Proof. (i)We take $p\in F$. Since $\{u_{in}\}$ is a bounded sequence in C for all $i\in I$, there exists $M>0$ such that

$$M=\max_{1\le i\le k}\{||u_{in}-p||\}.$$

Let $L = \max_{1 \le i \le k}\{L_i\} > 0$. Using (1.2), we have

$$\begin{aligned}
||y_{1n} - p|| &= ||P(\alpha_{1n}x_n + \beta_{1n}T_1(PT_1)^{n-1}x_n + \gamma_{1n}u_{1n}) - p|| \\
&\le ||\alpha_{1n}x_n + \beta_{1n}T_1(PT_1)^{n-1}x_n + \gamma_{1n}u_{1n} - (\alpha_{1n} + \beta_{1n} + \gamma_{1n})p|| \\
&\le \alpha_{1n}||x_n - p|| + \beta_{1n}||T_1(PT_1)^{n-1}x_n - p|| + \gamma_{1n}||u_{1n} - p|| \\
&\le \alpha_{1n}||x_n - p|| + \beta_{1n}L||x_n - p|| + \gamma_{1n}M \\
&\le (1 + L)||x_n - p|| + M.
\end{aligned}$$

Assume that

$$||y_{in} - p|| \le (1 + L)^i||x_n - p|| + \sum_{j=0}^{i-1} L^j M$$

holds for some $1 \le i \le k - 1$. Then

$$\begin{aligned}
||y_{(i+1)n} - p|| &= ||P(\alpha_{(i+1)n}x_n + \beta_{(i+1)n}T_{(i+1)}(PT_{(i+1)})^{n-1}y_{in} + \gamma_{(i+1)n}u_{(i+1)n}) - p|| \\
&\le \alpha_{(i+1)n}||x_n - p|| + \beta_{(i+1)n}||T_{(i+1)}(PT_{(i+1)})^{n-1}y_{in} - p|| + \gamma_{(i+1)n}||u_{(i+1)n} - p|| \\
&\le \alpha_{(i+1)n}||x_n - p|| + \beta_{(i+1)n}L||y_{in} - p|| + \gamma_{(i+1)n}||u_{(i+1)n} - p|| \\
&\le \alpha_{(i+1)n}||x_n - p|| + \beta_{(i+1)n}L\left[(1 + L)^i||x_n - p|| + \sum_{j=0}^{i-1} L^j M\right] + \gamma_{(i+1)n}||u_{(i+1)n} - p|| \\
&\le \left[\alpha_{(i+1)n} + \beta_{(i+1)n}L(1 + L)^i\right]||x_n - p|| + \beta_{(i+1)n}L\sum_{j=0}^{i-1} L^j M + \gamma_{(i+1)n}M \\
&\le [1 + L(1 + L)^i]||x_n - p|| + \sum_{j=1}^{i} L^j M + M \\
&\le (1 + L)^{i+1}||x_n - p|| + \sum_{j=0}^{i} L^j M
\end{aligned}$$

Therefore, by induction, we get for all $i \in I$

$$||y_{in} - p|| \le (1 + L)^i||x_n - p|| + \sum_{j=0}^{i-1} L^j M.$$

Now, from (1.2), it implies that

$$\begin{aligned}
||x_{n+1} - p|| &= ||P(\alpha_{kn}x_n + \beta_{kn}T_k(PT_k)^{n-1}y_{(k-1)n} + \gamma_{kn}u_{kn}) - p|| \\
&\le \alpha_{kn}||x_n - p|| + \beta_{kn}L||y_{(k-1)n} - p|| + \gamma_{kn}||u_{kn} - p|| \\
&\le \alpha_{kn}||x_n - p|| + \beta_{kn}L\left[(1 + L)^{k-1}||x_n - p|| + \sum_{j=0}^{k-2} L^j M\right] + \gamma_{kn}M \\
&\le \left[\alpha_{kn} + \beta_{kn}L(1 + L)^{k-1}\right]||x_n - p|| + \beta_{kn}L\sum_{j=0}^{k-2} L^j M + \gamma_{kn}M \\
&\le [1 + \theta_n(1 + L)^k]||x_n - p|| + \theta_n\left[L\sum_{j=0}^{k-2} L^j M + M\right] \\
&\le [1 + \theta_n M_0]||x_n - p|| + \theta_n M_1,
\end{aligned}$$

where $\theta_n = \beta_{kn} + \gamma_{kn}$, and $M_0 = (1 + L)^k$, and $M_1 = \sum_{j=0}^{k-1} L^j M$.

(ii) It is well known that $1 + x \le e^x$ for all $x \ge 0$. Estimate (2.1) yields

$$\begin{aligned}
\|x_{n+m} - p\| &\le (1 + \theta_{n+m-1}M_0)\|x_{n+m-1} - p\| + \theta_{n+m-1}M_1 \\
&\le e^{\theta_{n+m-1}M_0}[(1 + \theta_{n+m-2})M_0\|x_{n+m-2} - p\| + \theta_{n+m-2}M_1] + \theta_{n+m-1}M_1 \\
&\le e^{(\theta_{n+m-1}+\theta_{n+m-2})M_0}\|x_{n+m-2} - p\| + e^{\theta_{n+m-1}M_0}M_1(\theta_{n+m-1} + \theta_{n+m-2}) \\
&\quad \cdots\cdots \\
&\le e^{M_0\Sigma_{j=1}^{\infty}\theta_j}\|x_n - p\| + e^{M_0\Sigma_{j=1}^{\infty}\theta_j} \cdot M_1 \cdot \sum_{j=n}^{n+m-1}\theta_j \\
&\le M_2\|x_n - p\| + M_1M_2\sum_{j=n}^{n+m-1}\theta_j,
\end{aligned}$$

where $M_2 = e^{M_0\Sigma_{j=1}^{\infty}\theta_j}$.

□

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Banach space X. Let* $\{T_i : i \in I\}$*:* $C \to X$ *be a finite family of nonself uniformly quasi-Lipschitzian mappings with* $L_i > 0$*, i.e.,*

$$\|T_i(PT_i)^{n-1}x - p_i\| \le L_i\|x - p_i\|,$$

for all $x \in C$ *and* $p_i \in F(T_i)$*,* $i \in I$*. Define the sequence* $\{x_n\}$ *as in (1.2) with* $\sum_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$*. Suppose that* $F = \bigcap_{i=1}^{k} Fix(T_i) \neq \emptyset$ *and closed. Then* $\{x_n\}$ *converges to a common fixed point of* $\{T_i : i \in I\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$*, where* $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. The necessity is obvious. Next, we will prove the sufficiency. It follows from Lemma 2.1 that $\{x_n\}$ is bounded. From Lemma 2.1 (i), we have

$$\begin{aligned}
\|y_{1n} - p\| &\le \alpha_{1n}\|x_n - p\| + \beta_{1n}L\|x_n - p\| + \gamma_{1n}\|u_{1n} - p\| \\
&\le \alpha_{1n}\|x_n - p\| + \beta_{1n}L\|x_n - p\| + \gamma_{1n}[\|u_{1n} - x_n\| + \|x_n - p\|] \\
&\le (1 + L)\|x_n - p\| + M',
\end{aligned}$$

where

$$M' = \max_{1\le i\le k}\{\|u_{in} - x_n\|\}.$$

The same to the proof of Lemma 2.1, we get that

$$\|x_{n+1} - p\| \le [1 + \theta_n M_0]\|x_n - p\| + \theta_n M_1',$$

where $\theta_n = \beta_{kn} + \gamma_{kn}$, and $M_0 = (1 + L)^k$, and $M_1' = \sum_{j=0}^{k-1} L^j M'$. Taking infimum over all p in F, we obtain

$$d(x_{n+1}, F) \le (1 + \theta_n M_0)d(x_n, F) + \theta_n M_1'.$$

Note that M_1' does not depend on p.

Since $\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}(\beta_{kn} + \gamma_{kn}) < \infty$ and Lemma 1.1, we get that $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, from $\liminf_{n\to\infty} d(x_n, F) = 0$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$.

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, for any $\varepsilon > 0$, there exists a constant N_0 such that for all $n \geq N_0$, we have

$$d(x_n, F) \leq \frac{\varepsilon}{4M_2} \text{ and } \sum_{i=N_0}^{\infty} \theta_i \leq \frac{\varepsilon}{4M_1M_2}.$$

In particular, there exists a $p_1 \in F$ and a constant $N_1 > N_0$, such that

$$||x_{N_1} - p_1|| \leq \frac{\varepsilon}{4M_2}.$$

It follows from (2.2) that when $n > N_1$, we have

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - p_1|| + ||x_n - p_1|| \\ &\leq 2M_2||x_{N_1} - p_1|| + M_1M_2\left(\sum_{j=N_1}^{n+m-1} \theta_j + \sum_{j=N_1}^{n-1} \theta_j\right) \\ &\leq 2M_2\frac{\varepsilon}{4M_2} + M_1M_2\left(\frac{\varepsilon}{4M_1M_2} + \frac{\varepsilon}{4M_1M_2}\right) = \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in closed convex subset of real Banach spaces. Clearly, $\{x_n\}$ converges to a point of C.

Suppose that $\lim_{n\to\infty} x_n = p \in C$. We notice that

$$|d(p, F) - d(x_n, F)| \leq ||x_n - p||,$$

for all $n \geq 1$. Since $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} d(x_n, F) = 0$, we conclude that $d(p, F) = 0$. Therefore, $p \in F$. □

Corollary 2.1. *Let C be a nonempty closed convex subset of a real Banach space X. Let $\{T_i : i \in I\}$: $C \to X$ be a finite family of nonself uniformly Lipschitzian mappings with $L_i > 0$. Define the sequence $\{x_n\}$ as in (1.2) with $\sum_{n=1}^{\infty} (\beta_{kn} + \gamma_{kn}) < \infty$. Suppose that $F = \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$ and closed. Then $\{x_n\}$ converges to a common fixed point of $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{||x - p||: p \in F\}$.*

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Banach space X. Let $\{T_i : i \in I\}$: $C \to X$ be a finite family of nonself asymptotically nonexpansive mappings (or nonself asymptotically quasi-nonexpansive mappings) with $\{k_{in}\}$. Define the sequence $\{x_n\}$ as in (1.2) with $\sum_{n=1}^{\infty} (\beta_{kn} + \gamma_{kn}) < \infty$. Suppose that $F = \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$ and closed. Then $\{x_n\}$ converges to a common fixed point of $\{T_i : i \in I\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{||x - p|| : p \in F\}$.*

Proof. Since for all $i \in I$, $\{k_{in}\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_{in} = 1$, there exists $L_i > 0$ such that $L_i = \sup_{n\geq 1}\{k_{in}\} < \infty$. Consequently, $\{T_i : i \in I\}$ is a finite family of nonself uniformly quasi-Lipschitzian mappings with $L_i > 0$. From Theorem 2.1, we get the desired result. □

Remark 2.1. (i) When $\{T_i : i \in I\}$ is a finite family of asymptotically nonexpansive self-mappings or asymptotically quasi-nonexpansive self-mappings with $\{k_{in}\}$, Corollary 2.2 also holds.

(ii) In Corollary 2.2, we remove the condition: "$\sum_{n=1}^{\infty}(k_{in}-1)<\infty$", which is required in many other article (see, e.g., [2,4-9]).

(iii) When considering iterative schemes with errors, many authors need the conditions: "$\sum_{n=1}^{\infty}\gamma_{in}<\infty, i \in I$", see for example [4-6]. But in Corollary 2.2, we only need the condition: "$\sum_{n=1}^{\infty}\theta_n<\infty$", where $\theta_n = \beta_{kn} + \gamma_{kn}$.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Banach space X. Let* $\{T_i : i \in I\}$: $C \to X$ *be a finite family of nonself uniformly quasi-Lipschitzian mappings with* $L_i > 0$. *Define the sequence* $\{x_n\}$ *as in (1.2) with* $\sum_{n=1}^{\infty}(\beta_{kn}+\gamma_{kn})<\infty$. *Suppose that* $F=\bigcap_{i=1}^{k}\mathrm{Fix}(T_i)\neq\emptyset$, *and closed. If for any given* $1 \le l \le k$,

(i) $\lim_{n\to\infty}||x_n - T_l x_n|| = 0$;

(ii) *there exists a constant* $\alpha > 0$ *such that* $|| x_n - T_l x_n || \ge \alpha d(x_n, F)$ *for all* $n \ge 1$.

Then $\{x_n\}$ *converges to a common fixed point of* $\{T_i : i \in I\}$.

Proof. From the conditions (i) and (ii), it implies that $\lim_{n\to\infty}d(x_n,F)=0$. Therefore, from the proof of Theorem 2.1, $\{x_n\}$ converges to a common fixed point of $\{T_i : i \in I\}$. □

Remark 2.2. A mapping $T: C \to X$ is said to be semi-compact, if for any sequence $\{x_n\} \in C$ such that $\lim_{n\to\infty}||x_n - Tx_n|| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Banach space X. Let* $\{T_i : i \in I\}$: $C \to X$ *be a finite family of nonself uniformly quasi-Lipschitzian mappings with* $L_i > 0$. *Define the sequence* $\{x_n\}$ *as in (1.2) with* $\sum_{n=1}^{\infty}(\beta_{kn}+\gamma_{kn})<\infty$. *Suppose that* $F=\bigcap_{i=1}^{k}F(T_i)\neq\emptyset$*and closed. If*

(i) $\lim_{n\to\infty}||x_n - T_i x_n|| = 0$ *for all* $1 \le i \le k$;

(ii) *for some* $1 \le l \le k$, T_l *is semi-compact.*

Then $\{x_n\}$ *converges to a common fixed point of* $\{T_i : i \in I\}$.

Proof. Since T_l is semi-compact and $\lim_{n\to\infty}||x_n - T_l x_n|| = 0$, there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x* \in C$. Consequently, we have

$$||x^* - T_i x^*|| = \lim_{n_j\to\infty}||x_{n_j} - T_i x_{n_j}|| = \lim_{n_j\to\infty}||x_{n_j} - T_l x_{n_j}|| = 0.$$

This implies that $x^* \in F$. From Theorem 2.1, it follows that

$$||x_{n+1} - x^*|| \le [1 + \theta_n M_0]||x_n - x^*|| + \theta_n M_1'.$$

Since $\sum_{n=1}^{\infty}\theta_n<\infty$, it implies from Lemma 1.1 that there exist a constant $b \ge 0$ such that

$$\lim_{n\to\infty}||x_n - x^*|| = b.$$

From $x_{n_j} \to x^*$, we know that $b = 0$, i.e., $x_n \to x^*$. Thus, $\{x_n\}$ converge to a common fixed point of $\{T_i : i \in I\}$.

□

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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