Al-Mezel and Khamsi *Fixed Point Theory and Applications* 2012, **2012**:57
http://www.fixedpointtheoryandapplications.com/content/2012/1/57

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A convergence result on random products of mappings in metric trees

Saleh Abdullah Al-Mezel[1*] and Mohamed Amine Khamsi[2]

* Correspondence: salmezel@kau.edu.sa
[1]Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Let X be a metric space and $\{T_1, ..., T_N\}$ be a finite family of mappings defined on $D \subset X$. Let $r : \mathbb{N} \to \{1,..., N\}$ be a map that assumes every value infinitely often. The purpose of this article is to establish the convergence of the sequence (x_n) defined by

$$x_0 \in D; \text{ and } x_{n+1} = T_{r(n)}(x_n), \quad \text{for all } n \geq 0.$$

In particular we prove Amemiya and Ando's theorem in metric trees without compactness assumption. This is the first attempt done in metric spaces. These type of methods have been used in areas like computerized tomography and signal processing.

Mathematics Subject Classification 2000: Primary: 06F30; 46B20; 47E10.

Keywords: computerized tomography, convex feasibility problem, convex programming, metric tree, nonexpansive mapping, projection algorithm, projective mapping, random product, signal processing, unrestricted iteration, unrestricted product

1. Introduction

Many problems in mathematics [1] and physical sciences [2-4] uses a techniques known as search for a common fixed point. Indeed, let X be a metric space and suppose $T_1,..., T_N$ are pairwise distinct self-mappings of some nonempty and closed subset D of X. Suppose further that the fixed point set, $\mathrm{Fix}(T_i) = \{x \in D; T_i(x) = x\}$, of each mapping T_i is nonempty and that $C = \mathrm{Fix}(T_1) \cap \cdots \cap \mathrm{Fix}(T_N) \neq \emptyset$. The aim is to find a common fixed point of these mappings. One frequently employed approach is the following.

Let r be a random mapping for $\{1,..., N\}$, i.e., a surjective mapping from $\mathbb{N}$ onto $\{1,..., N\}$ that takes each value in $\{1,...,N\}$ infinitely often. Then generate a random sequence (x_n) by $x_0 \in D$ arbitrary, and

$$x_{n+1} = T_{r(n)}(x_n), \quad \textit{for all } n > 0,$$

and hope that this sequence converges to a point in C. We call it a random or unrestricted product (resp. iteration). For products generated by using control sequence, there are many results: for instance, cyclic control arises when $r(n) = n + 1 \mod N$ (see, for example, [5]).

In general, this random product fails to have well convergence behavior. The first positive results were done in the case when $D = X$ is a Hilbert space and each mapping

T_i, is the projection onto some nonempty, closed and convex subset C_i of X ; hence Fix $(T_i) = C_i$, $i = 1, ..., N$. The problem of finding a common fixed point is then the well-known *convex feasibility problem* (see, for example, [5]). Combettes article [6] proposed several interesting applications of this problem. Some of the early known results in this case are.

(1) Amemiya and Ando [7]: If each set C_i, is a closed subspace, then the random product converges weakly to the projection onto C.

(2) Bruck [8]: If some set C_i, is compact, then the random product converges in norm to a point in C. If $N = 3$ and each set C_i, is symmetric, then the random product converges weakly to a point in C (see also [[9], Theorem 2]).

(3) Aharoni and Censor [10], Flam and Zowe [11], Tseng [12], Eisner et al. [13]: If X is finite dimensional, then the random product converges in norm to a point in C.

The authors in [14] were successful in their extension of Amemiya and Ando's [7] results from Hilbert spaces to Banach spaces. In this study, we investigate Amemiya and Ando's result and [14]'s results in metric spaces. Such extension is the first attempt so far.

The main difficulty faced in such extensions is the heavy use of the linearity structure of either the Hilbert space in [7] or Banach spaces in [14]. Indeed when one tries to extend concepts from linear functional analysis, one has to pay attention to look deep into the supporting basic ideas and what intrinsic interrelations exist between them. Most of the main theorems in nonlinear functional analysis were done in the framework of linear Banach spaces. So it was interesting to investigate the extension of these fundamental results in nonlinear structures like metric spaces. As an example of this research is Kirk's fixed point theorem [15]. Many researchers have tried to do it but the best approach is the one given by Penot [16]. The impact of this approach went beyond what it intended to do initially. This research follows the same motivations. In particular we investigate the concept of weak convergence in metric spaces which is central, for instance, in [14]. We consider the case of metric trees to illustrate some of these ideas.

2. Basic definitions and results

Metric trees were first introduced by Tits [17] in 1977. A metric tree is a metric space (M, d) such that for every x, y in M there is a unique arc between x and y and this arc is isometric to an interval in $\mathbb{R}$. For example, a connected graph without loop is a metric tree. One basic property of metric trees is their one dimensionality. Again in the late seventies, while studying t-RNA molecules of the *E. Coli* bacterium Eigen raised several questions which led Dress [18,19] to construct metric trees, (named as T-theory). Metric trees also arise naturally in the study of group isometries of hyperbolic spaces. For metric properties of trees we refer to [20].

Since a metric tree is a space in which there is only one path between two points x and y, this would imply that if z is a point between x and y, by which we mean if $d(x, z) + d(z, y) = d(x, y)$ then we know that z is actually on the path between x and y. This will motivate the next concept of a metric interval.

Definition 2.1. *A metric interval or metric segment* $[x, y]$ *in a metric space M is defined by*

$$[x, y] := \{z \in M : d(x, z) + d(z, y) = d(x, y)\}.$$

First let us give the definition of a metric tree.

Definition 2.2. *A metric tree is a nonempty metric space M satisfying:*

(a) *Any two points x and y in M are the endpoints of a metric segment* $[x, y]$.

(b) *If* $x, y, z \in M$ *then* $[x, y] \cap [x, z] = [x, w]$ *for some* $w \in M$. *(i.e., if we have two metric segments with a common endpoint, then their intersection is a metric segment.)*

(c) *If* $x, y, z \in M$ *and* $[x, y] \cap [y, z] = \{y\}$ *then* $[x, y] \cup [y, z] = [x, z]$ *(i.e., If two metric segments intersect in a single point, then their union is a metric segment.)*

Metric trees are very special. They enjoy properties which are shared by l^∞ and Hilbert space. In particular, Kirk [21] showed that complete metric trees are hyperconvex. Since the weak topology has an intimate relationship with convexity, let us define convex subset in this setting.

Definition 2.3. *Let M be a metric tree and* $C \subset M$. *We say that C is convex if for all* $x, y \in C$ *we have* $[x, y] \subset C$.

Clearly a metric tree M and the empty set $\emptyset$ are convex. Also any closed ball $B(a, r) = \{z \in M: d(a, z) \leq r\}$ in a metric tree is also convex. Let $\mathcal{C}(M)$ denotes the collection of all closed and convex subsets of M, we set:

$$\text{conv}(A) = \cap \{B : B \text{ is a convex subset of } M \text{ such that } A \subseteq B\}.$$

Note that $\mathcal{C}(M)$ is invariant by intersection, i.e. the intersection of any family of convex subsets of M is convex. We need the following result of Baillon [22] in order to prove our first fact about $\mathcal{C}(M)$.

Theorem 2.1. [22]*Let M be a bounded metric space and let* $\{H_\beta\}_{\beta \in \Gamma}$ *be a decreasing family of nonempty hyperconvex subsets of M then* $\bigcap_{\beta \in \Gamma} H_\beta \neq \emptyset$ *and is hyperconvex.*

Since convex subsets of a metric tree are metric trees, then they are hyperconvex by [21]. This combined with Baillon's result we get the following theorem.

Theorem 2.2. *Let M be a bounded complete metric tree and let* $\{C_\beta\}_{\beta \in \Gamma}$ *be a family of nonempty, closed and convex subsets of M such that* $\bigcap_{\beta \in \Gamma_f} C_\beta \neq \emptyset$, *where* Γ_f *is any finite subset of* Γ, *then* $\bigcap_{\beta \in \Gamma} C_\beta \neq \emptyset$ *and is convex.*

This is known as compactness of $\mathcal{C}(M)$ according to Penot's formulation [16]. Note the slight difference between the statements of the two theorems. Indeed the intersection of two convex sets is convex while the intersection of two hyperconvex sets may not be hyperconvex.

Next we discuss the nearest point projections in metric trees. Let C be a nonempty, closed and convex subset of a complete metric tree M. For any $x \in M$, denote

$$P_C(x) = \left\{c \in C;\ d(x, c) = \text{dist}(x, C) = \inf_{y \in C} d(x, y)\right\}.$$

In a Hilbert space, the metric projections on closed and convex subsets are nonexpansive. In uniformly convex spaces, the metric projections are uniformly Lipschitzian. In fact, they are nonexpansive if and only if the space is Hilbert. In what follows we will show that the metric projections in metric trees are nonexpansive. This result is not true in hyperconvex metric spaces.

Lemma 2.1. [23,24]*If C is a nonempty, closed and convex subset of a complete metric tree M, then for any $x \in M$ there exists a unique $c_x \in C$ such that $dist(x,C) = d(x,c_x)$, which means that P_C is single valued. Moreover if $c \in C$ we have*

$$d(x, P_C(x)) + d(P_C(x), c) = d(x, c),$$

and

$$d(P_C(x), P_C(y)) = d(x, y) - d(x, P_C(x)) - d(y, P_C(y))$$

or $P_C(x) = P_C(y)$, for any $x, y \in M$. In particular, P_C is nonexpansive.

Next we prove another property of the mapping P_C.

Proposition 2.1. *If C is a nonempty, closed and convex subset of a complete metric tree M, then for any $x \in M$ we have*

$$P_C([x, P_C(x)]) = \{P_C(x)\}.$$

In other words, P_C is a sunny nonexpansive mapping [25,26].

Proof. Let $x \in M$ and $y \in [x,P_C(x)]$. Since M is a metric tree, there exists $w \in M$ such that $[y, P_C(x)] \cap [y, P_C(y)] = [y, w]$. We have $w \in [P_C(x), P_C(y)]$. Since C is convex, we get that $w \in C$. Also the definition of w implies

$$d(y, w) + d(w, P_C(y)) = d(y, P_C(y)).$$

The properties of P_C will force $P_C(y) = w$ which will imply $P_C(y) \in [y, P_C(x)]$. Since $y \in [x, P_C(x)]$ we get

$$d(x, P_C(x)) = d(x, y) + d(y, P_C(x)) = d(x, y) + d(y, P_C(y)) + d(P_C(y), P_C(x))$$

In particular we get $d(x, P_C(y)) \le d(x, P_C(x))$ which implies $P_C(y) = P_C(x)$.

3. Amemiya and Ando's theorem in metric trees

In 1965 Amemiya and Ando [7] proved the astonishing result.

Theorem 3.1. [7]*Let H be a Hilbert space and $\{P_1,..., P_N\}$ be a finite family of orthogonal linear projections defined on H. Let $r : \mathbb{N} \to \{1,..., N\}$ be a map that assumes every value infinitely often. The sequence (x_n) defined by*

$$x_0 \in H, \quad \text{and } x_{n+1} = P_{r(n)}(x_n), \quad \text{for all } n \geq 0.$$

converges weakly in H.

Today, 46 years later, it is still not known whether (x_n) converges strongly, even for $N = 3$. There is doubt expressed in the literature as to whether this sequence does converge strongly (cf. [[27], Example 4]) for an interesting example of possible relevance. In general, strong convergence may be obtained when some kind of compactness is assumed. Next, we show that in the case of metric trees, we have strong convergence without any compactness assumption. The Amemiya and Ando's theorem was preceded by von Neumann [28] for alternating products of two projections (with strong convergence as the conclusion).

Theorem 3.2. *Let $C_1, \ldots, C_N$ be a finite family of nonempty, closed and convex subsets of a complete metric tree M such that* $C = \bigcap_{1 \le i \le N} C_i \neq \emptyset$. *Let $r : \mathbb{N} \to \{1, \ldots, N\}$ be a map that assumes every value infinitely often. The sequence (x_n) defined by*

$$x_0 \in X, \text{ and } x_{n+1} = P_{C_{r(n+1)}}(x_n), \quad \text{for all } n \geq 0.$$

converges strongly in M. Moreover we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} P_{C_{r(n+1)}} \circ \cdots \circ P_{C_{r(1)}}(x_0) = P_C(x_0).$$

Fix

Proof. $c \in C$. Using Lemma 2.1 we have

$$d(x_n, x_{n+1}) = d(x_n, c) - d(x_{n+1}, c),$$

for any $n \geq 0$. In particular we have

$$\sum_{k=n}^{k=n+h} d(x_k, x_{k+1}) = d(x_n, c) - d(x_{n+h}, c),$$

for any $n \geq 0$ and $h \geq 0$. Since $P_{C_{r(n)}}(c) = c$, we get $d(x_{n+1},c) \leq d(x_n, c)$, for any $n \geq 0$. In other words, the sequence $(d(x_n,c))$ is decreasing. Hence $\lim_{n\to\infty} d(x_n, c)$ exists. Therefore the sequence (x_n) is Cauchy. Since M is complete, there exists $c \in M$ such that $\lim_{n\to\infty} x_n = c$. For any $i \in \{1,\ldots, N\}$, there exists a subsequence of (x_n) which belongs to C_i. Since C_i is closed, we conclude that $c \in C_i$, for any $i \in \{1,\ldots, N\}$. Hence $c \in C$. Next we show that $c = P_C(x_0)$. For any $n \geq 0$, we get

$$d(x_0, x_n) \leq \sum_{k=0}^{k=n} d(x_k, x_{k+1}) = d(x_0, c_0) - d(x_n, c_0).$$

If we let $n \to \infty$, we get

$$d(x_0, c) \leq d(x_0, c_0) - d(c, c_0).$$

The definition of c_0 implies

$$d(x_0, c) \leq d(x_0, c_0) - d(c, c_0) \leq d(x_0, c) - d(c, c_0)$$

which implies $d(c, c_0) = 0$, or $c = c_0$.

Remark 3.1. *In* [14]*the authors made heavy use of the property that in smooth reflexive Banach spaces X, if E is a closed subspace of X, then there is at most one nonexpansive retraction of X onto E* [26]. *In the case of metric trees, we have a similar result. Indeed, let C be a nonempty, closed and convex subset of a metric tree M. Then P_C is a sunny nonexpansive retract of M onto C. Let $Q : M \to C$ be another sunny mapping. Let $x \in M$. There exists $w \in M$ such that $[x, P_C(x)] \cap [x, Q(x)] = [x, w]$. Since C is convex, then $w \in C$. Also since $d(x,w) + d(w, P_C(x)) = d(x, P_C(x))$, the definition of $P_C(x)$ will force $P_C(x) = w$. Hence $P_C(x) \in [x, Q(x)]$. Since Q is sunny, we must have $Q(P_C(x)) = Q(x)$, which implies $P_C(x) = Q(x)$. In other words, P_C is the only sunny retract from M onto C.*

In the next section we investigate the behavior of the random product of mappings other than the nearest point projections.

4. Random product of mappings in metric trees

As the authors did in [14], one inspires itself from the Amemiya and Ando's work in Hilbert spaces to extend it to other underlying spaces. In particular the authors in [14] introduced the concepts of (*W*) and (*S*) properties. Since the (*W*) property is strongly linked to the weak-topology, we are not able to extend such property to metric trees.

Definition 4.1. *Let M be a metric space. Let* $T : M \to M$ *be a nonexpansive map with a nonempty fixed point set Fix*(*T*). *We will say that T satisfies the property (S) if and only if for any* $c \in Fix(T)$ *and any sequence* (v_n) *such that* $\lim_{n\to\infty}\left[d(v_n, c) - d(T(v_n), c)\right] = 0$, *we have* $\lim_{n\to\infty} d(v_n, T(v_n)) = 0$.

Remark 4.1. *Note that if* $T_1,\ldots,T_N$ *are nonexpansive mappings with a common fixed point and satisfy the property (S), then we have*

$$\mathrm{Fix}(T_1 \circ \cdots \circ T_N) = \bigcap_{1\le i\le N} \mathrm{Fix}(T_i).$$

Indeed let $c_0 \in M$ *be a common fixed point of* $T_1, \ldots, T_N$. *Let us only prove that* $Fix(T_1 \circ \cdots \circ T_N) \subset \bigcap_{1\le i\le N} Fix(T_i)$. *Let* $c \in Fix(T_1 \circ \ldots \circ T_N)$. *Then we have* $T_1 \circ \ldots \circ T_N(c) = c$.

Since each mapping is nonexpansive, we get

$$d(c_0, c) = d(c_0, T_1 \circ \cdots \circ T_N(c)) \le d(c_0, T_N(c)) \le d(c_0, c).$$

Since T_N *satisfies the property* (*S*), *we get* $T_N(c) = c$. *Similarly one will show that* $T_i(c) = c$, *for* $i = 1,\ldots,N$.

Another property discovered by Caristi [29] (see also [30]) and extensively used to obtain some beautiful results extending Banach contraction principle is the following definition.

Definition 4.2. *Let M be a metric space. Let* $T : M \to M$ *be a mapping. We will say that T satisfies the property (C)-*λ *if and only if there exists a map* $\lambda : M \to [0, \infty)$ *such that*

$$d(x, T(x)) \le \lambda(x) - \lambda(T(x)),$$

for any $x \in M$.

It is easy to check that if *T* satisfies the (C)-λ property, then any orbit $(T^n(x))$ is a Cauchy sequence for any $x \in M$. In particular if *M* is complete and *T* is continuous, then $P(x) = \lim_{n\to\infty} T^n(x)$ is a retraction from *M* into Fix(*T*) which is nonempty.

Example 4.1. *Let* $M = [0,2]$ *is a metric tree being an interval of the metric tree* $\mathbb{R}$. *Define the mapping* $T : [0,2] \to [0,1]$ *by*

$$T(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2. \end{cases}$$

Note that T is a nonexpansive retraction and $Fix(T) = [0,1]$. *In particular* $(T^n(x))$ *is convergent and its limit is T*(*x*). *But the nearest point projection on* $[0,1]$ *is the map*

$$P(x) = \begin{cases} x \text{ if } 0 \le x \le 1 \\ 1 \text{ if } 1 \le x \le 2 \end{cases}$$

which is different from T. Moreover one can easily show that

$$d(x, T(x)) \leq \lambda(x) - \lambda(T(x))$$

for any $x \in [0,2]$, *where* $\lambda(t) = t$. *Therefore T satisfies the (C)-λ property.*

In the next result, we show how Theorem 3.2 extends to the family of mappings satisfying the (C)-λ property.

Theorem 4.1. *Let M be a complete metric space. Let* $T_1,...,T_N$ *be a finite family of pairwise distinct self-mappings of some nonempty and closed subset D of M. Suppose further that each map* T_i, $i = 1,..., N$, *is continuous and satisfies the (C)-λ property, with the same function λ. Let r be a random mapping for* $\{1,...,N\}$, *i.e., a surjective mapping from* $\mathbb{N}$ *onto* $\{1,...,N\}$ *that takes each value in* $\{1,...,N\}$ *infinitely often. Then the random sequence* (x_n) *defined by* $x_0 \in D$ *arbitrary, and*

$$x_{n+1} = T_{r(n)}(x_n), \quad \text{for all } n > 0,$$

is convergent. Its limit is a common fixed point of the mappings $T_1,...,T_N$.

Proof. Let $x \in D$. Our assumptions on the mappings T_i imply

$$d(x_n, x_{n+1}) \leq \lambda(x_n) - \lambda(x_{n+1}),$$

for any $n \geq 0$. In particular we have

$$\sum_{k=n}^{k=n+h} d(x_k, x_{k+1}) \leq \lambda(x_n) - \lambda(x_{n+h}),$$

for any $n \geq 0$ and $h \geq 0$. On the other hand, we have $\lambda(x_{n+1}) \leq \lambda(x_n)$, for any $n \geq 0$. Therefore the positive sequence $(\lambda(x_n))$ is convergent. Clearly this will imply that the sequence (x_n) is Cauchy. Since M is complete, there exists $c \in M$ such that $\lim_{n\to\infty} x_n = c \in D$ since D is closed. For any $i \in \{1,..., N\}$, there exists a subsequence of $(x_{\varphi(n)})$ such that $x_{\varphi(n)+1} = T_i(x_{\varphi(n)})$. Hence $T_i(c) = c$, for any $i = 1,..., N$.

Note that the limit defines a retraction on the common fixed point set of the mappings $T_1,...,T_N$. But this retraction may not be equal to the nearest point projection even in the case of a metric tree as the Example 4.1 shows.

The next result investigates the extension of some of the results discovered in [14]. Before we do this, we need to discuss the weak-topology in the nonlinear setting of metric spaces. Indeed, let (x_n) be a bounded sequence in the metric tree M. Define the real-valued function

$$\varphi_{\mathcal{U}}(x) = \lim_{n,\mathcal{U}} d(x_n, x)$$

where $\mathcal{U}$ is a nontrivial ultrafilter [31]. We have the following theorem which will play a central role in our work.

Theorem 4.2. *Let M be a complete metric tree. Let* (x_n) *be a bounded sequence in M. Then for any nontrivial ultrafilter* $\mathcal{U}$, *there exists a unique* $z_{\mathcal{U}} \in M$ *such that*

$$\varphi_{\mathcal{U}}(x) = \varphi_{\mathcal{U}}(z_{\mathcal{U}}) + d(x, z_{\mathcal{U}}),$$

for any $x \in M$.

Proof. Let

$$r = \inf\{\varphi_{\mathcal{U}}(x);\ x \in M\}.$$

For any $\varepsilon > 0$, consider the set

$$C_\varepsilon = \{x \in M;\ \varphi_{\mathcal{U}}(x) \leq r + \varepsilon\}.$$

Using the properties of metric trees, we know that C_ε is a nonempty, bounded and convex subset of M. Since $\varphi_{\mathcal{U}}$ is continuous, then it is also closed. Using the compactness of $\mathcal{C}(M)$, then

$$\bigcap_{\varepsilon>0} C_\varepsilon = \{x \in M; \varphi_{\mathcal{U}}(x) = r\} \neq \emptyset.$$

Now, we will show that this intersection is reduced to one point. Indeed, let us Fix $z_{\mathcal{U}} \in M$ such that $\varphi_{\mathcal{U}}(z_{\mathcal{U}}) = r$. Let x be any point in M. Using the properties of metric trees, for any $n \geq 1$, there exists $w_n \in [x, z_{\mathcal{U}}]$ such that $[x_n, x] \cap [x_n, z_{\mathcal{U}}] = [x_n, w_n]$. Since $[x, z_{\mathcal{U}}]$ is compact, then there exists $w \in [x, z_{\mathcal{U}}]$ such that $\lim_{\mathcal{U}} d(w_n, w) = 0$. Since $d(x_n, w_n) + d(w_n, z_{\mathcal{U}}) = d(x_n, z_{\mathcal{U}})$ for any $n \geq 1$ then we have

$$\lim_{\mathcal{U}} d(x_n, w_n) + \lim_{\mathcal{U}} d(w_n, z_{\mathcal{U}}) = \lim_{\mathcal{U}} d(x_n, z_{\mathcal{U}}) = \varphi_{\mathcal{U}}(z_{\mathcal{U}}).$$

Hence

$$\lim_{\mathcal{U}} d(x_n, w) + d(w, z_{\mathcal{U}}) = \varphi_{\mathcal{U}}(z_{\mathcal{U}}),$$

that is, $\varphi_{\mathcal{U}}(w) + d(w, z_{\mathcal{U}}) = \varphi_{\mathcal{U}}(z_{\mathcal{U}})$. Obviously this will imply $d(w, z_{\mathcal{U}}) = 0$ or $\lim_{\mathcal{U}} d(w_n, z_{\mathcal{U}}) = 0$. Also since $d(x_n, w_n) + d(w_n, x) = d(x_n, x)$ for any $n \geq 1$, then we have

$$\lim_{\mathcal{U}} d(x_n, w_n) + \lim_{\mathcal{U}} d(w_n, x) = \lim_{\mathcal{U}} d(x_n, z_{\mathcal{U}}) = \varphi_{\mathcal{U}}(x).$$

Hence

$$\lim_{\mathcal{U}} d(x_n, z_{\mathcal{U}}) + d(z_{\mathcal{U}}, x) = \varphi_{\mathcal{U}}(x),$$

that is, $\varphi_{\mathcal{U}}(z_{\mathcal{U}}) + d(z_{\mathcal{U}}, x) = \varphi_{\mathcal{U}}(x)$. This latest identity, also known as Uniform Opial condition, will easily show that $z_{\mathcal{U}}$ is unique.

Definition 4.3. *Let M be a metric tree and (x_n) be a bounded sequence in M. For any nontrivial ultrafilter $\mathcal{U}$, the unique point $z_{\mathcal{U}}$found in Theorem* 4.2 *is called the weak-limit of (x_n) along $\mathcal{U}$. We will say that (x_n) is weakly convergent if and only if $z_{\mathcal{U}} = z_{\mathcal{V}}$, for any nontrivial ultrafilters $\mathcal{U}$and $\mathcal{V}$.*

It is because of the absence of a dual space that we used Opial behavior to try to catch the weak-limit of a bounded sequence. In the next result we show some close similarities between the classical weak-limit point in Banach spaces and the one introduced above.

Proposition 4.1. *Let M be a complete metric tree and (x_n) be a bounded sequence in M. For any nontrivial ultrafilter $\mathcal{U}$, then*

$$z_{\mathcal{U}} \in \Omega(x_n) = \bigcap_{n\geq 1} \overline{\text{conv}}\{x_n, x_{n+1}, \ldots\}.$$

Fix

Proof. $i \geq 1$. Set P_n the nearest point projection on $\overline{\text{conv}}(x_i)_{i\geq n}$. Since $\mathcal{U}$ is nontrivial then

$$\{i; d(x_i, P_n(z_{\mathcal{U}})) \leq d(x_i, z_{\mathcal{U}})\} \in \mathcal{U}$$

where we used the nonexpansiveness of P_n. This obviously implies

$$\lim_{i,\mathcal{U}} d(x_i, P_n(z_{\mathcal{U}})) \leq \lim_{i,\mathcal{U}} d(x_i, z_{\mathcal{U}})$$

which implies $P_n(z_{\mathcal{U}}) = z_{\mathcal{U}}$ or $z_{\mathcal{U}} \in \overline{\text{con}}(x_i)_{i\geq n}$ for any $n \geq 1$. So $z_{\mathcal{U}} \in \Omega(x_n)$.

Next we discuss the behavior of mapping which satisfies the property (S).

Theorem 4.3. *Let M be a complete metric tree. Let $T : C \to C$ be a nonexpansive mapping which satisfies the property (S), where C is a nonempty, bounded, closed, and convex subset of M. Then the sequence $(T^n(x))$ converges weakly to a fixed point of T.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be any nontrivial ultrafilters. Let $z_{\mathcal{U}}$ and $z_{\mathcal{V}}$ be the minimum point of $\varphi_{\mathcal{U}}(z) = \lim_{\mathcal{U}} d(T^n(x), z)$ and $\varphi_{\mathcal{V}}(z) = \lim_{\mathcal{V}} d(T^n(x), z)$, respectively. Proposition 4.1 implies that $z_{\mathcal{U}}$ and $z_{\mathcal{V}}$ are in $\Omega(T^n(x)) \subset C$. Next we will prove that $z_{\mathcal{U}}$ and $z_{\mathcal{V}}$ are fixed point of T. It is enough to prove that $T(z_{\mathcal{U}}) = z_{\mathcal{U}}$. Since C is hyperconvex and bounded, we know that T has a nonempty fixed point set (see [32,33]). Let $c \in \text{Fix}(T)$. The sequence $(d(T^n(x),c))$ is a decreasing sequence of positive numbers. Since T satisfies the property (S), we deduce that $\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0$. Hence

$$\varphi_{\mathcal{U}}(z) = \lim_{\mathcal{U}} d(T^n(x), z) = \lim_{\mathcal{U}} d(T^{n+1}(x), z),$$

which implies $\varphi_{\mathcal{U}}(T(z)) \leq \varphi_{\mathcal{U}}(z)$, for any $z \in C$. The properties of $z_{\mathcal{U}}$ will force the identity $T(z_{\mathcal{U}}) = z_{\mathcal{U}}$, i.e., $z_{\mathcal{U}} \in \text{Fix}(T)$. Note that the sequence $(d(T^n(x), z_{\mathcal{U}}))$ is decreasing which implies

$$\varphi_{\mathcal{U}}(z_{\mathcal{U}}) = \lim_{\mathcal{U}} d(T^n(x), z_{\mathcal{U}}) = \lim_{n\to\infty} d(T^n(x), z_{\mathcal{U}}) = \inf_{n\geq 1} d(T^n(x), z_{\mathcal{U}}).$$

Hence

$$\varphi_{\mathcal{U}}(z_{\mathcal{V}}) = \lim_{\mathcal{U}} d(T^n(x), z_{\mathcal{V}}) = \lim_{n\to\infty} d(T^n(x), z_{\mathcal{V}}) = \lim_{\mathcal{V}} d(T^n(x), z_{\mathcal{V}}),$$

because $z_V \in \text{Fix}(T)$. Since

$$\varphi_{\mathcal{V}}(z_{\mathcal{V}}) = \lim_{\mathcal{V}} d(T^n(x), z_{\mathcal{V}}) \leq \varphi_{\mathcal{V}}(z_{\mathcal{U}}) = \lim_{\mathcal{V}} d(T^n(x), z_{\mathcal{U}}) = \lim_{n\to\infty} d(T^n(x), z_{\mathcal{U}}),$$

which implies $\varphi_{\mathcal{U}}(z_{\mathcal{V}}) \leq \varphi_{\mathcal{U}}(z_{\mathcal{U}})$. The properties of $z_{\mathcal{U}}$ imply $z_{\mathcal{V}} = z_{\mathcal{U}}$. This completes the proof of Theorem 4.3.

Acknowledgement
The authors gratefully acknowledge the financial support provided by the University of Tabuk through the project of international cooperation with the University of Texas at El Paso. The authors thank the referee for pointing out some oversights and calling attention to some related literature.

Author details
[1]Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia [2]Department of Mathematical Sciences, The University of Texas at El Paso El Paso, TX 79968, USA

Authors' contributions
All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 21 September 2011 Accepted: 13 April 2012 Published: 13 April 2012

References

1. Deutsch, F: The method of alternating orthogonal projections. In: Singh SP (ed.) Approximation Theory Spline Features and Applications. pp. 105–121. Kluwer Academic, Amsterdam (1992). Proc. Conf., Hotel Villa del Mare, Maratea, Italy, 28 April 1991-9 May 1991
2. Censor, Y: Parallel application of block-iterative methods in medical imaging and radiation therapy. Math Program. **42**, 307–325 (1988)
3. Censor, Y, Herman, GT: On some optimization techniques in image reconstruction from projections. Appl Numer Math. **3**, 365–391 (1987)
4. Sezan, MI: An overview of convex projections theory and its applications to image recovery problems. Ultramicroscopy. **40**, 55–67 (1992)
5. Bauschke, HH, Borwein, JM: On projection algorithms for solving convex feasibility problems. Siam Rev. **38**(3):367–426 (1996)
6. Combettes, PL: Hilbertian convex feasibility problem: convergence of projection methods. Appl Math Optim. **35**, 311–330 (1997)
7. Amemiya, I, Ando, T: Convergence of random products of contractions in Hilbert space. Acta Sci Math (Szeged). **26**, 239–244 (1965)
8. Bruck, RE: Random products of contractions in metric and Banach spaces. J Math Anal Appl. **88**, 319–332 (1982)
9. Dye, J, Reich, S: Unrestricted iterations of nonexpansive mappings in Hilbert space. Nonlinear Anal. **18**, 199–207 (1992)
10. Aharoni, R, Censor, Y: Block-iterative projection methods for parallel computation of solutions to convex feasibility problems. Linear Algebra Appl. **120**, 165–175 (1989)
11. Flam, SD, Zowe, J: Relaxed outer projections, weighted averages and convex feasibility. BIT. **30**, 289–300 (1990)
12. Tseng, P: On the convergence of the products of firmly nonexpansive mappings. SIAM J Optim. **2**, 425–434 (1992)
13. Eisner, L, Koltracht, I, Neumann, M: Convergence of sequential and asynchronous nonlinear para-contractions. Numer Math. **62**, 305–319 (1992)
14. Dye, JM, Khamsi, MA, Reich, S: Random products of contractions in Banach spaces. Trans Am Math Soc. **325**(1):87–99 (1991)
15. Kirk, WA: A fixed point theorem for mappings which do not increase distances. Am Math Month. **72**, 1004–1006 (1965)
16. Penot, JP: Fixed point theorems without convexity. Bull Soc Math France. **60**, 129–152 (1979)
17. Tits, J: A Theorem of Lie-Kolchin for Trees. Contributions to Algebra: a Collection of Papers Dedicated to Ellis Kolchin. Academic Press, New York (1977)
18. Dress, AWM: Trees, tight extensions of metric spaces, and the chomological dimension of certain groups: a note on combinatorial properties of metric spaces. Adv Math. **53**, 321–402 (1984)
19. Dress, AWM, Moulton, V, Terhalle, W: *T*-theory, an overview. Eur J Combin. **17**, 161–175 (1996)
20. Buneman, P: A note on the metric properties of trees. J Combin Theory Ser B. **17**, 48–50 (1974)
21. Kirk, WA: Hyperconvexity of ℝ-trees. Fund Math. **156**, 67–72 (1998)
22. Baillon, JB: Nonexpansive mappings and hyperconvex spaces. Contem Math. **72**, 11–19 (1988)
23. Aksoy, AG, Khamsi, MA: Fixed points of uniformly Lipschitzian mappings in metric trees. Scientiae Mathematicae Japonicae. **65**, 31–41 (2007)
24. Aksoy, AG, Khamsi, MA: A selection theorem in metric trees. Proc Am Math Soc. **134**, 2957–2966 (2006)
25. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker New York (1984)
26. Reich, S: Product formulas, nonlinear semigroups, and accretive operators. J Funct Anal. **36**(2):147–168 (1980)
27. Dye, J, Reich, S: On the unrestricted iteration of projections in Hilbert space. J Math Anal Appl. **156**, 101–119 (1991)
28. von Neumann, J: Functional operators Vol II. In Ann Math Stud, vol. 22,Princeton University Press (1950). This is a reprint of mimeographed lecture notes first distributed in 1933
29. Caristi, J: Fixed point theorems for mappings satisfying inwardness conditions. Trans Am Math Soc. **215**, 241–251 (1976)
30. Ekeland, I: Sur les problemes variationnels. Comptes Rendus Acad Sci Paris. **275**, 1057–1059 (1972)
31. Aksoy, AG, Khamsi, MA: Nonstandard Methods in Fixed Point Theory. pp. 139. Springer-Verlag, New York/Berlin (1990)
32. Sine, R: Hyperconvexity and nonexpansive multifunctions. Trans Am Math Soc. **315**, 755–767 (1989)
33. Soardi, P: Existence of fixed points of nonexpansive mappings in certain Banach lattices. Proc Am Math Soc. **73**, 25–29 (1979)

doi:10.1186/1687-1812-2012-57
Cite this article as: Al-Mezel and Khamsi: A convergence result on random products of mappings in metric trees. *Fixed Point Theory and Applications* 2012 **2012**:57.