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Shrinking projection algorithms for equilibrium problems with a bifunction defined on the dual space of a Banach space

Jia-wei Chen¹, Yeol Je Cho^{2*} and Zhongping Wan¹

* Correspondence: yjcho@gnu.ac.kr
²Department of Mathematics
Education and the RINS,
Gyeongsang National University,
Chinju 660-701, Republic of Korea
Full list of author information is
available at the end of the article

Abstract

Shrinking projection algorithms for finding a solution of an equilibrium problem with a bifunction defined on the dual space of a Banach space, in this paper, are introduced and studied. Under some suitable assumptions, strong and weak convergence results of the shrinking projection algorithms are established, respectively. Finally, we give an example to illustrate the algorithms proposed in this paper.

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1 Introduction

Let Ω be a nonempty closed subset of a real Hilbert space H . Let g be a bifunction from $\Omega \times \Omega$ to R , where R is the set of real numbers. *The equilibrium problem* for g is as follows: Find $\bar{x} \in \Omega$ such that

$$g(\bar{x}, \gamma) \geq 0, \quad \forall \gamma \in \Omega.$$

Many problems in structural analysis, optimization, management sciences, economics, variational inequalities and complementary problems coincide to find a solution of the equilibrium problem. Various methods have been proposed to solve some kinds of equilibrium problems in Hilbert and Banach spaces (see [1-8]).

In [9], Takahashi and Zembayashi proved strong and weak convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in Banach spaces. Ibaraki and Takahashi [10] introduced a new resolvent of a maximal monotone operator in Banach spaces and the concept of the generalized nonexpansive mapping in Banach spaces. Honda et al. [11], Kohsaka and Takahashi [12] also studied some properties for the generalized nonexpansive retractions in Banach spaces. Takahashi et al. [13] proved a strong convergence theorem for nonexpansive mapping by hybrid method. In 2009, Ceng et al. [2] proved strong and weak convergence theorems for equilibrium problems and dealt maximal monotone operators by hybrid proximal-point methods. Motivated by Ibaraki

and Takahashi [10] and Takahashi et al. [13], Takahashi and Zembayashi [14] considered the following equilibrium problem:

Let E be a smooth Banach space with dual space E^* and C be a nonempty closed subset of E such that $J(C)$ is a closed and convex subset of E^* , where J is the normalized duality mapping from E onto E^* . Let $f: J(C) \times J(C) \rightarrow R$ be a mapping. Consider the equilibrium problem as follows: Find $\bar{x} \in C$ such that

$$f(J(\bar{x}), J(\gamma)) \geq 0, \quad \forall \gamma \in C. \tag{1.1}$$

Then they proved a strong convergence theorem for finding a solution of the equilibrium problem (1.1) in Banach spaces. Forward, we denote the set of solutions of the problem (1.1) by $EP(f)$:

Inspired and motivated by Ceng et al. [2], Takahashi and Zembayashi [14], Takahashi and Zembayashi [9], the main aim of this paper is to introduce and investigate a new iterative method for finding a solution of the equilibrium problem (1.1). Under some appropriate assumptions, strong and weak convergence results of the iterative algorithms are established, respectively. Furthermore, we also give an example to illustrate the algorithms proposed in this paper.

2 Preliminaries

Throughout this paper, we denote the sets of nonnegative integers and real numbers by Z_+ and R , respectively.

Let E be a real Banach space with the dual space E^* . The norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. The weak convergence and strong convergence are denoted by \rightharpoonup and \rightarrow , respectively. Let C be a nonempty closed subset of E . We denote the *normalized duality mapping* from E to E^* by J defined by

$$J(x) = \{j(x) \in E^* : \langle j(x), x \rangle = \|j(x)\| \|x\| = \|j(x)\|^2 = \|x\|^2\}, \quad \forall x \in E.$$

J is said to be *weakly sequentially continuous* if the strong convergence of a sequence $\{x_n\}$ to x in E implies the weak* convergence of $\{J(x_n)\}$ to $J(x)$ in E^* .

Many properties of the normalized duality mapping J can be found in [15-17] and, now, we list the following properties:

- (p¹) $J(x)$ is nonempty for any $x \in E$;
- (p²) J is a monotone and bounded operator in Banach spaces;
- (p³) J is a strictly monotone operator in strictly convex Banach spaces;
- (p⁴) J is the identity operator in Hilbert spaces;
- (p⁵) If E is a reflexive, smooth and strictly convex Banach space and $J^*: E^* \rightarrow 2^E$ is the normalized duality mapping on E^* , then $J^{-1} = J^*$; $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_{E^*} and I_E are the identity mappings on E^* and E , respectively.
- (p6) If E is a strictly convex Banach space, then J is one to one, that is,

$$x \neq \gamma \quad \Rightarrow \quad J(x) \cap J(\gamma) = \emptyset;$$

- (p⁷) If E is smooth, then J is single-valued;
- (p⁸) E is a uniformly convex Banach space if and only if E^* is uniformly smooth;
- (p⁹) If E is uniformly convex and uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded subsets of E^* :

Let E be a smooth Banach space. Let a function $\phi: E \times E \rightarrow R$ be defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Then we have

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, J(z) - J(y) \rangle, \quad \forall x, y, z \in E.$$

Remark 2.1. (see [17,18]) The following statements hold:

(1) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$, $\phi(x; y) = 0$ if and only if $x = y$;

(2) If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$;

(3) For all $x, y \in E$, $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$.

For solving the equilibrium problem (1.1), we assume that $f: J(C) \times J(C) \rightarrow R$ satisfies the following conditions (A1) - (A4) [9]:

(A1) $f(x^*, x^*) = 0$ for all $x^* \in J(C)$;

(A2) f is monotone, that is, $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in J(C)$;

(A3) f is upper hemicontinuous, that is, for all $x^*, y^*, z^* \in J(C)$,

$$\limsup_{t \rightarrow 0^+} f(x^* + t(z^* - x^*), y^*) \leq f(x^*, y^*);$$

(A4) For all $x^* \in J(C)$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

In the sequel, we recall some concepts and results.

Definition 2.1. (see [11]) Let C be a nonempty closed subset of a smooth Banach space E . A mapping $T: C \rightarrow C$ is said to be *generalized nonexpansive* if $F(T)$ is nonempty and

$$\phi(Tx, p) \leq \phi(x, p), \quad \forall x \in C, p \in F(T),$$

where $F(T)$ denotes the set of fixed points of T , that is, $F(T) = \{x \in C: Tx = x\}$.

Definition 2.2. (see [11]) Let C be a nonempty closed subset of E . A mapping $R: E \rightarrow C$ is called:

(1) a *retraction* if $R^2 = R$;

(2) *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t > 0$.

Definition 2.3. (see [11]) A nonempty closed subset C of a smooth Banach space E is called a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction R from E onto C .

Lemma 2.1. (see [19]) Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \rightarrow 0$.

Lemma 2.2. (see [18]) Let E be a uniformly convex Banach space. Then, for any $r > 0$; there exists a strictly increasing, continuous and convex function $h: [0, 2r] \rightarrow R$ such that $h(0) = 0$ and

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)h(\|x-y\|), \quad \forall x, y \in B_r, t \in [0, 1],$$

where $B_r = \{z \in E: \|z\| \leq r\}$.

Lemma 2.3. (see [1]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that $J(C)$ is closed and convex. Assume that a mapping $f: J(C) \times J(C) \rightarrow R$ satisfies the conditions (A1)-(A4). Then, for any $r > 0$ and $x \in$

E , there exists $z \in C$ such that

$$f(J(z), J(y)) + \frac{1}{r} \langle z - x, J(y) - J(z) \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4. (see [14]) Let C be a nonempty closed subset of a uniformly smooth, strictly convex and reflexive Banach space E such that $J(C)$ is closed and convex. Assume that a mapping $f: J(C) \times J(C) \rightarrow R$ satisfies the conditions (A1)-(A4). For any $r > 0$ and $x \in E$, define a mapping $T_r: E \rightarrow C$ by

$$T_r(x) = \left\{ z \in C : f(J(z), J(y)) + \frac{1}{r} \langle z - x, J(y) - J(z) \rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in E.$$

Then the following statements hold:

- (1) T_r is single-valued;
- (2) For all $x, y \in E$,

$$\langle T_r(x) - T_r(y), J(T_r(x)) - J(T_r(y)) \rangle \leq \langle x - y, J(T_r(x)) - J(T_r(y)) \rangle;$$

- (3) $F(T_r) = EP(f)$ and $J(EP(f))$ is closed and convex;
- (4) $\varphi(x, T_r(x)) + \varphi(T_r(x), p) \leq \varphi(x, p)$ for all $x \in E$ and $p \in F(T_r)$.

Lemma 2.5. (see [9]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E , and let R be a retraction of E onto C . Then the following statements are equivalent:

- (1) R is sunny generalized nonexpansive;
- (2) $\langle x - Rx, J(y) - J(Rx) \rangle \leq 0$ for all $(x, y) \in E \times C$.

Lemma 2.6. (see [20]) Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.

Lemma 2.7. (see [10]) Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C . Then, for any $x \in E$ and $z \in C$, the following statements hold:

- (1) $z = Rx$ if and only if $\langle x - z, J(y) - J(z) \rangle \leq 0$ for all $y \in C$;
- (2) $\varphi(x, Rx) + \varphi(Rx, z) \leq \varphi(x, z)$.

Lemma 2.8. (see [12]) Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E . Then the following statements are equivalent:

- (1) C is a sunny generalized nonexpansive retract of E ;
- (2) $J(C)$ is closed and convex.

Remark 2.2. If E is a Hilbert space, then, from Lemmas 2.6 and 2.8, a sunny generalized nonexpansive retraction from E onto C reduces to a metric projection operator P from E onto C .

Lemma 2.9. (see [12]) Let C be a nonempty closed sunny generalized nonexpansive retract subset of a smooth, strictly convex and reflexive Banach space E . Let R be the sunny generalized nonexpansive retraction from E onto C . Then, for any $x \in E$ and $z \in C$,

$$z = Rx \iff \phi(x, z) = \min_{y \in C} \phi(x, y).$$

Lemma 2.10. (see [21]) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \quad \forall n \in \mathbb{Z}_+.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

3 Main results

In this section, we propose iterative algorithms for finding a solution of the equilibrium problem (1.1) and prove the strong and weak convergence for the algorithms in a Banach space under some suitable conditions.

Theorem 3.1. Let C be a nonempty closed subset of a uniformly convex and uniformly smooth Banach space E such that $J(C)$ is closed and convex. Assume that a mapping $f: J(C) \times J(C) \rightarrow R$ satisfies the conditions (A1)-(A4). Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C, \\ u_n \in C \text{ such that } f(J(u_n), J(y)) + \frac{1}{r_n} \langle u_n - x_n, J(y) - J(u_n) \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)u_n), \quad \forall n \in \mathbb{Z}_+, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ such that

$$\sum_{n=0}^{\infty} \alpha_n < \infty, \quad \liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0, \quad \liminf_{n \rightarrow \infty} r_n > 0.$$

Then the sequence $\{R_{EP(f)}x_n\}$ converges strongly to a point $\omega \in EP(f)$, where $R_{EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $EP(f)$.

Proof. For the sake of simplicity, let $u_n = T_{r_n}x_n$ and $y_n = \beta_n x_n + (1 - \beta_n)u_n$. Then $x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)y_n$. From Lemma 2.4, it follows that $EP(f)$ is a nonempty closed and convex subset of E .

First, we claim that $\{x_n\}$ is bounded. Indeed, let $\omega \in EP(f)$. Since

$$\begin{aligned} \phi(y_n, \omega) &= \|\beta_n x_n + (1 - \beta_n)u_n\|^2 - 2\langle \beta_n x_n + (1 - \beta_n)u_n, J(\omega) \rangle + \|\omega\|^2 \\ &\leq \beta_n \|x_n\|^2 + (1 - \beta_n)\|u_n\|^2 - 2\beta_n \langle x_n, J(\omega) \rangle - 2(1 - \beta_n)\langle u_n, J(\omega) \rangle + \|\omega\|^2 \\ &= \beta_n \phi(x_n, \omega) + (1 - \beta_n)\phi(u_n, \omega) \\ &= \beta_n \phi(x_n, \omega) + (1 - \beta_n)\phi(T_{r_n}x_n, \omega) \\ &\leq \phi(x_n, \omega), \end{aligned}$$

we have

$$\begin{aligned} \phi(x_{n+1}, \omega) &\leq \alpha_n \phi(x_0, \omega) + (1 - \alpha_n)\phi(y_n, \omega) \\ &\leq \alpha_n \phi(x_0, \omega) + (1 - \alpha_n)\phi(x_n, \omega) \\ &\leq \alpha_n \phi(x_0, \omega) + \phi(x_n, \omega). \end{aligned}$$

By virtue of $\sum_{n=0}^{\infty} \alpha_n < \infty$ and Lemma 2.10, it follows that the limit of $\{\phi(x_n, \omega)\}$ exists. Therefore, $\{\phi(x_n, \omega)\}$ is bounded and so $\{x_n\}, \{u_n\}$ and $\{y_n\}$ are also bounded. Let $z_n = R_{EP(f)}x_n$. Then $z_n \in EP(f)$ and so, from Lemma 2.7, we have

$$\phi(x_n, z_n) = \phi(x_n, R_{EP(f)}x_n) \leq \phi(x_n, \omega) - \phi(R_{EP(f)}x_n, \omega) \leq \phi(x_n, \omega).$$

Therefore, $\{z_n\}$ is bounded and so $\phi(x_0, z_n)$ is bounded. Since $\phi(x_{n+1}, z_n) \leq \alpha_n \phi(x_0, z_n) + \phi(x_n, z_n)$, by Lemma 2.7, one has

$$\begin{aligned} \phi(x_{n+1}, z_{n+1}) &= \phi(x_{n+1}, R_{EP(f)}x_{n+1}) \\ &\leq \phi(x_{n+1}, z_n) - \phi(R_{EP(f)}x_{n+1}, z_n) \\ &\leq \phi(x_{n+1}, z_n) \\ &\leq \alpha_n \phi(x_0, z_n) + \phi(x_n, z_n). \end{aligned}$$

Since $\{\phi(x_0, z_n)\}$ is bounded, there exists $M > 0$ such that $|\phi(x_0, z_n)| \leq M$. By $\sum_{n=0}^{\infty} \alpha_n < \infty$, we have

$$\sum_{n=0}^{\infty} \alpha_n \phi(x_0, z_n) \leq M \sum_{n=0}^{\infty} \alpha_n < \infty,$$

that is, $\sum_{n=0}^{\infty} \alpha_n \phi(x_0, z_n) < \infty$. From Lemma 2.10, it follows that $\{\phi(x_n, z_n)\}$ is a convergent sequence. For any $m \in \mathbb{Z}_+ \setminus \{0\}$, one can get

$$\phi(x_{n+m}, \omega) \leq \phi(x_n, \omega) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, \omega).$$

Then we have

$$\phi(x_{n+m}, z_n) \leq \phi(x_n, z_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n).$$

From $z_{n+m} = R_{EP(f)}x_{n+m}$ and Lemma 2.7, it follows that

$$\phi(x_{n+m}, z_{n+m}) + \phi(z_{n+m}, z_n) \leq \phi(x_{n+m}, z_n) \leq \phi(x_n, z_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n)$$

and so

$$\phi(z_{n+m}, z_n) \leq \phi(x_n, z_n) - \phi(x_{n+m}, z_{n+m}) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n).$$

Set $r = \sup\{\|z_n\|; n \in \mathbb{Z}_+\}$. Then, from Lemma 2.2 and [19], it follows that there is a strictly increasing, continuous and convex function $h: [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and

$$h(\|z_n - z_{n+m}\|) \leq \phi(z_{n+m}, z_n) \leq \phi(x_n, z_n) - \phi(x_{n+m}, z_{n+m}) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_0, z_n).$$

Since $\{\phi(x_n, z_n)\}$ is convergent, $\{\phi(x_0, z_n)\}$ is bounded and $\sum_{n=0}^{\infty} \alpha_n$ is convergent, it follows that, for any $m \in \mathbb{Z}_+$,

$$\lim_{n \rightarrow \infty} \|z_n - z_{n+m}\| = 0,$$

which shows that $\{z_n\}$ is a Cauchy sequence. Since $EP(f)$ is closed, there exists $\omega \in EP(f)$ such that $z_n \rightarrow \omega$. Therefore, the sequence $\{R_{EP(f)}x_n\}$ converges strongly to the $\omega \in EP(f)$. This completes the proof. \square

Theorem 3.2. Let C be a nonempty closed subset of a uniformly convex and uniformly smooth Banach space E such that $J(C)$ is closed and convex. Assume that a

mapping $f: J(C) \times J(C) \rightarrow R$ satisfies the conditions (A1)-(A4). Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C, \\ u_n \in C \text{ such that } f(J(u_n), J(y)) + \frac{1}{r_n} \langle u_n - x_n, J(y) - J(u_n) \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)u_n), \quad \forall n \in Z_+, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ such that

$$\sum_{n=0}^{\infty} \alpha_n < \infty, \quad \liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0, \quad \liminf_{n \rightarrow \infty} r_n > 0.$$

If J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to a point $\omega \in EP(f)$, where $\omega = \lim_{n \rightarrow \infty} R_{EP(f)} x_n$ and $R_{EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $EP(f)$.

Proof. For the sake of simplicity, let $u_n = T_{r_n} x_n$, $y_n = \beta_n x_n + (1 - \beta_n)u_n$ and $z_n = R_{EP(f)} x_n$. As in the proof of Theorem 3.1, we have $\{x_n\}, \{u_n\}, \{z_n\}, \{J(x_n)\}$ and $\{y_n\}$ are bounded. Set $r = \sup\{\|x_n\|, \|z_n\|: n \in Z_+\}$. It follows from Lemma 2.2 that there exists a strictly increasing, continuous and convex function $h: [0, 2r] \rightarrow R$ such that $h(0) = 0$ and

$$\|\beta_n x_n + (1 - \beta_n)u_n\|^2 \leq \beta_n \|x_n\|^2 + (1 - \beta_n) \|u_n\|^2 - \beta_n(1 - \beta_n)h(\|x_n - u_n\|).$$

Since

$$\begin{aligned} \phi(y_n, \omega) &= \phi(\beta_n x_n + (1 - \beta_n)u_n, \omega) \\ &\leq \beta_n \|x_n\|^2 + (1 - \beta_n) \|u_n\|^2 - \beta_n(1 - \beta_n)h(\|x_n - u_n\|) \\ &\quad - 2\beta_n \langle x_n, J(\omega) \rangle - 2(1 - \beta_n) \langle u_n, J(\omega) \rangle + \|\omega\|^2 \\ &= \beta_n \phi(x_n, \omega) + (1 - \beta_n) \phi(u_n, \omega) - \beta_n(1 - \beta_n)h(\|x_n - u_n\|) \\ &= \beta_n \phi(x_n, \omega) + (1 - \beta_n) \phi(T_{r_n} x_n, \omega) - \beta_n(1 - \beta_n)h(\|x_n - u_n\|) \\ &\leq \phi(x_n, \omega) - \beta_n(1 - \beta_n)h(\|x_n - u_n\|), \end{aligned}$$

we have

$$\begin{aligned} \phi(x_{n+1}, \omega) &= \phi(\alpha_n x_0 + (1 - \alpha_n)y_n, \omega) \\ &\leq \alpha_n \phi(x_0, \omega) + (1 - \alpha_n) \phi(y_n, \omega) \\ &\leq \alpha_n \phi(x_0, \omega) + \phi(y_n, \omega) \\ &\leq \alpha_n \phi(x_0, \omega) + \phi(x_n, \omega) - \beta_n(1 - \beta_n)h(\|x_n - u_n\|). \end{aligned}$$

Moreover, one has

$$\beta_n(1 - \beta_n)h(\|x_n - u_n\|) \leq \phi(x_n, \omega) - \phi(x_{n+1}, \omega) + \alpha_n \phi(x_0, \omega).$$

From $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, $\sum_{n=0}^{\infty} \alpha_n < \infty$ and the limit existence of $\{\phi(x_n, \omega)\}$, we have

$$\lim_{n \rightarrow \infty} h(\|x_n - u_n\|) = 0.$$

By the property of h , we get

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on the bounded subset of E , we obtain

$$\lim_{n \rightarrow \infty} \|J(x_n) - J(u_n)\| = 0.$$

Since $\{J(x_n)\}$ is bounded, we have that $J(x_n) \rightarrow p^*$ (here we may take a subnet $\{x_{n_k}\}$ of $\{x_n\}$ if necessary). Then $J(u_n) \rightarrow p^*$. From $\liminf_{n \rightarrow \infty} r_n > 0$, it follows that $\lim_{n \rightarrow \infty} \frac{\|x_n - u_n\|}{r_n} = 0$. Note that

$$f(J(u_n), J(y)) + \frac{1}{r_n} \langle u_n - x_n, J(y) - J(u_n) \rangle \geq 0.$$

By (A2), we obtain

$$f(J(y), J(u_n)) \leq -f(J(u_n), J(y)) \leq \frac{1}{r_n} \langle u_n - x_n, J(y) - J(u_n) \rangle.$$

Therefore, it follows that $f(J(y), p^*) \leq 0$. Let $\gamma_t^* = tJ(y) + (1-t)p^*$ for any $t \in (0,1)$. Then $\gamma_t^* \in J(C)$. Since

$$0 = f(\gamma_t^*, \gamma_t^*) \leq tf(\gamma_t^*, J(y)) + (1-t)f(\gamma_t^*, p^*) \leq tf(\gamma_t^*, J(y)),$$

we get $f(\gamma_t^*, J(y)) \geq 0$. By (A3), one has $f(p^*, J(y)) \geq 0$. Therefore, $p^* \in J(EP(f))$.

Let $z_n = R_{EP(f)}x_n$. From Theorem 3.1, one can get that $z_n \rightarrow \omega$ and so

$$\langle x_n - z_n, p^* - J(z_n) \rangle \leq 0.$$

Since J is weakly sequentially continuous, we have

$$\langle J^{-1}(p^*) - J^{-1}(J(\omega)), J(\omega) - p^* \rangle \geq 0. \tag{3.1}$$

By the monotonicity of J^{-1} ,

$$\langle J^{-1}(p^*) - J^{-1}(J(\omega)), J(\omega) - p^* \rangle \leq 0. \tag{3.2}$$

Thus, from both (3.1) and (3.2), it follows that

$$\langle J^{-1}(p^*) - J^{-1}(J(\omega)), J(\omega) - J(J^{-1}(p^*)) \rangle = 0,$$

this together with the strictly monotonicity of J yields that $J^{-1}(p^*) = \omega$. Therefore, the sequence $\{x_n\}$ converges weakly to the point $\omega \in EP(f)$, where $\omega = \lim_{n \rightarrow \infty} R_{EP(f)}x_n$. This completes the proof. \square

4 Numerical test

In this section, we give an example of numerical test to illustrate the algorithms given in Theorems 3.1 and 3.2.

Example 4.1. Let $E = R$, $C = [-1000, 1000]$ and define $f(x, y) = -5x^2 + xy + 4y^2$. Find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

First, we verify that f satisfies the conditions (A1)-(A4) as follows:

(A1) $f(x, x) = -5x^2 + x^2 + 4x^2 = 0$ for all $x \in [-1000, 1000]$;

(A2) $f(x, y) + f(y, x) = -(x - y)^2 \leq 0$ for all $x, y \in [-1000, 1000]$;

(A3) For all $x, y, z \in [-1000, 1000]$,

$$\begin{aligned} \limsup_{t \rightarrow 0^+} f(x + t(z - x), y) &= \limsup_{t \rightarrow 0^+} -5((1 - t)x + tz)^2 + (1 - t)xy + tzy + 4y^2 \\ &= -5x^2 + xy + 4y^2 \\ &\leq f(x, y). \end{aligned}$$

(A4) For all $x \in [-1000, 1000]$, $F(y) = f(x, y) = -5x^2 + xy + 4y^2$ is convex and lower semicontinuous.

From Lemma 2.4, T_r is single-valued. Now, we deduce a formula for $T_r(x)$. For any $y \in C, r > 0$,

$$f(z, y) + \frac{1}{r}(z - x, y - z) \geq 0 \Leftrightarrow 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2 \geq 0.$$

Set $G(y) = 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4r, b = (r + 1)z - x$ and $c = xz - (5r + 1)z^2$. So its discriminant $\Delta = b^2 - 4ac$ is

$$\begin{aligned} \Delta &= [(r + 1)z - x]^2 - 16r(xz - (5r + 1)z^2) \\ &= (r + 1)^2z^2 - 2(r + 1)xz + x^2 - 16rxz + (80r^2 + 16r)z^2 \\ &= [(9r + 1)z - x]^2. \end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C$, this is true if and only if $\Delta \leq 0$. That is, $[(9r + 1)z - x]^2 \leq 0$. Therefore, $z = \frac{x}{9r + 1}$, which yields that $T_r(x) = \frac{x}{9r + 1}$. Let $r_n = \frac{n}{n + 1}, \beta_n = \frac{n}{3n + 1}$ and $\alpha_n = \frac{1}{(3n + 1)^2}$. It is easy to check that

$$\sum_{n=0}^{\infty} \alpha_n < +\infty, \quad \liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) = \frac{2}{9} > 0, \quad \liminf_{n \rightarrow \infty} r_n = 1.$$

Thus, from Lemma 2.4, it follows that $EP(f) = \{0\}$. Therefore, all the assumptions in Theorems 3.1 and 3.2 are satisfied. Setting $x_0 = 1$ and using the algorithm in Theorem 3.1, we obtain the following sequences:

$$\begin{cases} x_0 = 1, \\ u_n = T_{r_n}(x_n) = \frac{n+1}{10n+1}x_n, \\ x_{n+1} = \frac{1}{(3n+1)^2}x_0 + \frac{108n^4+108n^3+33n^2+6n}{270n^4+297n^3+117n^2+19n+1}x_n. \end{cases}$$

Therefore, by Theorem 3.1, the sequence $\{P_{EP(f)} x_n\}$ must converge strongly to a solution of the problem (4.1). In fact, $P_{EP(f)} x_n = 0$ for all $n \in \mathbb{Z}_+$. Also, according to Theorem 3.2, the sequence $\{x_n\}$ converges weakly to a solution of the problem (4.1). For a number $\varepsilon = 10^{-3}$, if we use MATLAB, then we generate a sequence $\{x_n\}$ as follows:

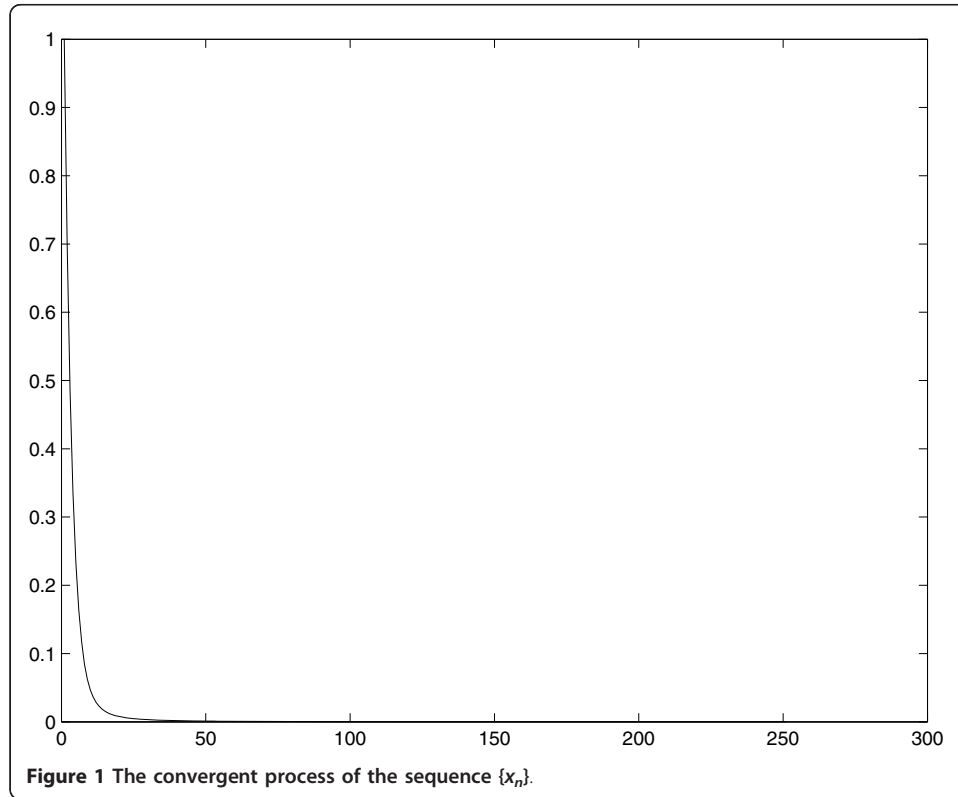
Selected values of $\{u_n\}$ and $\{x_n\}$ computed by computer programs are listed below Tables 1 and 2, respectively. The convergent process of the sequence $\{x_n\}$ is described in Figure 1.

Table 1 Selected values of $\{u_n\}$

u_n	u_n	u_n	u_n	u_n	u_n
0.1818	0.0607	0.0026	0.0012	0.0007	0.0003
0.0002	0.0001	0.0001	0.0001	0.0000	0.0000

Table 2 Selected values of $\{x_n\}$

x_n	x_n	x_n	x_n	x_n	x_n
0.4247	0.0204	0.0100	0.0059	0.0028	0.0021
0.0016	0.0013	0.0010	0.0008	0.0007	0.0006
0.0005	0.0004	0.0003	0.0003	0.0002	0.0002
0.0002	0.0002	0.0002	0.0001	0.0001	0.0001
0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
0.0001	0.0001	0.0001	0.0000	0.0000	0.0000



From Table 1, we can see that the sequence $\{u_n\}$ converges to 0. Moreover, $F(T_r) = EP(f) = \{0\}$. Table 2 shows that the iterative sequence $\{x_n\}$ converges to 0, which is indeed a solution of the problem (4.1). Moreover, $\lim_{n \rightarrow \infty} P_{EP(f)}x_n = 0$.

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Author details

¹School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, China ²Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea

Authors' contributions

J-WC, YJC and ZW carried out the studies on nonlinear analysis and applications, wrote this article together and participated in its design of this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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