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Weak convergence theorem for the three-step iterations of non-Lipschitzian nonself mappings in Banach spaces

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Abstract

In this article, we introduce a new three-step iterative scheme for the mappings which are asymptotically nonexpansive in the intermediate sense in Banach spaces. Weak convergence theorem is established for this three-step iterative scheme in a uniformly convex Banach space that satisfies Opial's condition or whose dual space has the Kadec-Klee property. Furthermore, we give an example of the nonself mapping which is asymptotically nonexpansive in the intermediate sense but not asymptotically nonexpansive. The results obtained in this article extend and improve many recent results in this area.

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Keywords: asymptotically nonexpansive in the intermediate sense non-self mapping, Kadec-Klee property, Opial?'?s condition, common fixed point

1 Introduction

Fixed-point iterations process for nonexpansive and asymptotically nonexpansive mappings in Banach spaces have been studied extensively by various authors [1-13]. In 1991, Schu [4] considered the following modified Mann iteration process for an asymptotically nonexpansive map T on C and a sequence $\{\alpha_n\}$ in [0, 1]:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 1. \tag{1.1}$$

Since then, Schu's iteration process (1.1) has been widely used to approximate fixed points of asymptotically nonexpansive mappings in Hilbert spaces or Banach spaces [7,8,10-13]. Noor, in 2000, introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces [6]. Later, Xu and Noor [7], Cho et al. [8], Suantai [9], Plubtieng et al. [12] studied the convergence of the three-step iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable.

In most of these articles, the operator T remains a self-mapping of a nonempty closed convex subset C of a uniformly convex Banach space X. If, however, the domain of T, $D(T)$, is a proper subset of X (and this is the case in several applications), and T maps $D(T)$ into X, then the iterative sequence $\{x_n\}$ may fail to be well defined. One method that has been used to overcome this is to introduce a retraction P. A subset C

of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$ and P is said to be a retraction. Recent results on approximation of fixed points of nonexpansive or asymptotically nonexpansive nonself mappings can be found in [14-19] and the references cited therein. For example, in 2003, Chidume et al. [16] introduced the following modified Mann iteration process and got the convergence theorems for asymptotically nonexpansive nonself-mapping:

$$x_1 \in C, \quad x_{n+1} = P[\alpha_n x_n + (1 - \alpha_n) T(PT)^{n-1} x_n], \quad n \geq 1. \tag{1.2}$$

Recently, Thianwan [18] generalized the iteration process (1.2) as follows: $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P[\alpha_n y_n + (1 - \alpha_n) T_1 (PT_1)^{n-1} y_n]; \\ y_n &= P[\beta_n x_n + (1 - \beta_n) T_2 (PT_2)^{n-1} x_n]. \end{aligned} \tag{1.3}$$

Obviously, if $\beta_n = 1$ for all $n \geq 1$, then (1.3) reduces to (1.2). Thianwan [18] proved the weak convergence theorem of the iteration process (1.3) in uniformly convex Banach spaces that satisfy Opial's condition.

The concept of asymptotically nonexpansive in the intermediate sense nonself mappings was introduced by Chidume et al. [20] as an important generalization of asymptotically nonexpansive in the intermediate sense self-mappings.

Definition 1.1 *Let C be a nonempty subset of a Banach space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and the following inequality holds:*

$$\limsup_{n \to +\infty} \sup_{x,y \in C} (\| T(PT)^{n-1}x - T(PT)^{n-1}y \| - \| x - y \|) \leq 0.$$

It should be noted that in [20-22], the asymptotically nonexpansive in the intermediate sense mapping is required to be uniformly continuous. In this article, we assume the continuity of T instead of uniform continuity. Chidume et al. [20] gave the weak convergence theorem for uniformly continuous nonself mapping which is asymptotically nonexpansive in the intermediate sense in uniformly convex Banach space whose dual space has the Kadec-Klee property.

Inspired and motivated by [16,18,20], we investigate the weak convergence theorem of three-step iteration process for continuous nonself mappings which are asymptotically nonexpansive in the intermediate sense in this article. Since the asymptotically nonexpansive in the intermediate sense mappings are non-Lipschitzian and Bruck's Lemma [23] do not extend beyond Lipschitzian mappings, new techniques are needed for this more general case. Utilizing the technique of the modulus of convexity and a new demiclosed principle for nonself-maps of Kazor [24], we establish the weak convergence theorem of the three-step iterative scheme in a uniformly convex Banach space that satisfies Opial's condition or whose dual space has the Kadec-Klee property, which extends and improves the recently announced ones in [4,16,18-20]. It should be noted that our theorems are new even in the case that the space has a Fréchet differentiable norm. In the end, to illustrate our theorem, we give a nonself mapping which is asymptotically nonexpansive in the intermediate sense but not asymptotically nonexpansive.

2 Preliminaries

Let X be a Banach space and X^* be its dual, then the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$ and we associate the set

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}.$$

It follows from the Hahn-Banach theorem that $J(x) \neq \emptyset$ for any $x \in X$. Then the multi-valued operator $J : X \mapsto X^*$ is called the normalized duality mapping of X. Recall that a Banach space X is said to be uniformly convex if for each $\varepsilon \in [0, 2]$, the modulus of convexity of X defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2} \| x + y \| : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \varepsilon\},$$

satisfies the inequality $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. Note that every closed convex subset of a uniformly convex Banach space is a retract. We say that X has the Kadec-Klee property if for every sequence $\{x_n\} \subset X$, whenever $x_n \rightharpoonup x$ with $||x_n|| \to ||x||$, it follows that $x_n \to x$. We would like to remark that a reflexive Banach space X with a Fréchet differentiable norm implies that its dual X^* has Kadec-Klee property, while the converse implication fails [25].

Recall that a Banach space X is said to satisfies Opial's condition if $x_n \rightharpoonup x$ and $x \neq y$ implies that

$$\limsup_{n\to+\infty} \| x_n - x \| < \limsup_{n\to+\infty} \| x_n - y \| .$$

The following lemmas are needed to prove our main results in next section.

Lemma 2.1 [5]*Let the nonnegative number sequences* $\{c_n\}$ *and* $\{w_n\}$ *satisfy*

$$c_{n+1} \leq c_n + w_n, \quad n \in N$$

If $\sum_{n=1}^{+\infty} w_n < +\infty$, then $\lim_{n\to+\infty} c_n$ *exists.*

Lemma 2.2 [4]*Suppose that X is a uniformly convex Banach space and for all positive integers n,* $0 < p \leq t_n \leq q < 1$. *If* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences of X such that* $\limsup_{n\to+\infty} \| y_n \| \leq r$, $\limsup_{n\to+\infty} \| y_n \| \leq r$ *and*

$$\lim_{n\to+\infty} \| t_n x_n + (1 - t_n) y_n \| = r$$

hold for some $r \geq 0$. *Then* $\lim_{n\to+\infty} \| x_n - y_n \| = 0$.

Lemma 2.3 [3]*Let X be a uniformly convex Banach space. If* $||x|| \leq 1$, $||y|| \leq 1$ *and* $||x - y|| \geq \varepsilon > 0$, *then for all* $\lambda \in [0, 1]$,

$$\| \lambda x + (1 - \lambda) y \| \leq 1 - 2\lambda(1 - \lambda)\delta(\varepsilon).$$

Lemma 2.4 [26]*Let X be a Banach space and J be the normalized duality mapping. Then for given* $x, y \in X$ *and* $j(x + y) \in J(x + y)$, *we have*

$$\| x + y\|^2 \leq \| x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 2.5 (Demiclosed principle for nonself-map [24]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and* $T : C \to X$ *be a nonself mapping which is continuous and asymptotically nonexpansive in the intermediate sense. If* $\{x_n\}$ *is a sequence in C converging weakly to x and*

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\| x_n - T(PT)^{k-1}x_n \|= 0,$$

then $x \in F(T)$, *i.e.,* $Tx = x$.

3 Main results

Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $P : X \to C$ be a nonexpansive retraction from X onto C. Let T_1, T_2, $T_3 : C \to X$ be three continuous nonself mappings which are asymptotically nonexpansive in the intermediate sense. Suppose that

$$r_n = \max\{0, \sup_{x,y\in C;i=1,2,3.} \| T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y \| - \| x-y \|\},$$

then $r_n \geq 0$, $\lim_{n\to+\infty} r_n = 0$ and for all $x, y \in C$ and $n \in N$,

$$\| T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y \| - \| x-y \|\leq r_n, \quad i = 1, 2, 3.$$

For a given $x_1 \in C$, we define the sequence $\{x_n\} \subset C$ by

$$\begin{aligned} x_{n+1} &= P[\alpha_n^{(1)}z_n + (1-\alpha_n^{(1)})T_1(PT_1)^{n-1}z_n]; \\ z_n &= P[\alpha_n^{(2)}y_n + (1-\alpha_n^{(2)})T_2(PT_2)^{n-1}y_n]; \\ y_n &= P[\alpha_n^{(3)}x_n + (1-\alpha_n^{(3)})T_3(PT_3)^{n-1}x_n]. \end{aligned} \tag{3.1}$$

where $\{\alpha_n^{(i)}\}$ is in 0[1] with $0 < p \leq \alpha_n^{(i)} \leq q < 1$, $i = 1, 2, 3$.

We also assume that the sequence $\{r_n\}$ satisfies $\sum_{n=1}^{+\infty} r_n < +\infty$ and the set of common fixed points of $\{T_i\}_{i=1}^3$ is nonempty, i.e.,

$$F = \cap_{i=1}^3 F(T_i) = \{x \in C : T_1x = T_2x = T_3x = x\} \neq \emptyset.$$

Lemma 3.1

$$\lim_{n\to+\infty} \| x_n - f \|= \lim_{n\to+\infty} \| y_n - f \|= \lim_{n\to+\infty} \| z_n - f \|= r \tag{3.2}$$

exists for all $f \in F$.

Proof. For all $f \in F$,

$$\begin{aligned} &\| y_n - f \|=\| P[\alpha_n^{(3)}x_n + (1-\alpha_n^{(3)})T_3(PT_3)^{n-1}x_n] - f \| \\ &\leq\| [\alpha_n^{(3)}x_n + (1-\alpha_n^{(3)})T_3(PT_3)^{n-1}x_n] - f \| \\ &\leq \alpha_n^{(3)} \| x_n - f \| +(1-\alpha_n^{(3)}) \| T_3(PT_3)^{n-1}x_n - f \| \\ &=\| x_n - f \| +r_n. \end{aligned}$$

Hence

$$\begin{aligned} &\| z_n - f \|=\| P[\alpha_n^{(2)}y_n + (1-\alpha_n^{(2)})T_2(PT_2)^{n-1}y_n] - f \| \\ &\leq\| [\alpha_n^{(2)}y_n + (1-\alpha_n^{(2)})T_2(PT_2)^{n-1}y_n] - f \| \\ &\leq\| y_n - f \| +r_n \\ &\leq\| x_n - f \| +2r_n. \end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned} \| x_{n+1} - f \| &= \| P[\alpha_n^{(1)} z_n + (1-\alpha_n^{(1)}) T_1 (PT_1)^{n-1} z_n] - f \| \\ &\le \| [\alpha_n^{(1)} z_n + (1-\alpha_n^{(1)}) T_1 (PT_1)^{n-1} z_n] - f \| \\ &\le \| z_n - f \| + r_n \\ &\le \| x_n - f \| + 3r_n. \end{aligned} \tag{3.4}$$

Put $w_n = 3r_n$, then we can obtain $\sum_{n=1}^{+\infty} w_n < +\infty$ and

$$\| x_{n+1} - f \| \le \| x_n - f \| + w_n.$$

By Lemma 2.1, we can conclude that

$$\lim_{n\to+\infty} \| x_n - f \| = r$$

exists. Combining it with (3.4), we have

$$\lim_{n\to+\infty} \| z_n - f \| = r.$$

Hence by (3.3), we get

$$\lim_{n\to+\infty} \| y_n - f \| = r.$$

This completes the proof.

Lemma 3.2

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \| x_n - T_i (PT_i)^{k-1} x_n \| = 0, \quad i = 1, 2, 3.$$

Proof. By (3.2) and (3.4), we can get

$$\begin{aligned} r &= \lim_{n\to+\infty} \| [\alpha_n^{(1)} z_n + (1-\alpha_n^{(1)}) T_1 (PT_1)^{n-1} z_n] - f \| \\ &= \lim_{n\to+\infty} \| (1-\alpha_n^{(1)}) [T_1 (PT_1)^{n-1} z_n - f] + \alpha_n^{(1)} (z_n - f) \| \end{aligned}$$

Then it follows from Lemma 2.2 and $\limsup_{n\to+\infty} \| T_1 (PT_1)^{n-1} z_n - f \| \le r$ that

$$\lim_{n\to+\infty} \| T_1 (PT_1)^{n-1} z_n - z_n \| = 0. \tag{3.5}$$

According to (3.3), we have

$$\begin{aligned} r &= \lim_{n\to+\infty} \| [\alpha_n^{(2)} y_n + (1-\alpha_n^{(2)}) T_2 (PT_2)^{n-1} y_n] - f \| \\ &= \lim_{n\to+\infty} \| (1-\alpha_n^{(2)}) [T_2 (PT_2)^{n-1} y_n - f] + \alpha_n^{(2)} (y_n - f) \| \end{aligned}$$

Noting $\limsup_{n\to+\infty} \| T_2 (PT_2)^{n-1} y_n - f \| \le r$, by Lemma 2.2 again, we can get

$$\lim_{n\to+\infty} \| T_2 (PT_2)^{n-1} y_n - y_n \| = 0. \tag{3.6}$$

Similarly, we can obtain

$$\lim_{n\to+\infty} \| T_3 (PT_3)^{n-1} x_n - x_n \| = 0. \tag{3.7}$$

Hence, it follows from

$$\begin{aligned}\| y_n - x_n \| &= \| P[\alpha_n^{(3)} x_n + (1-\alpha_n^{(3)}) T_3(PT_3)^{n-1} x_n] - x_n \| \\ &\le \| [\alpha_n^{(3)} x_n + (1-\alpha_n^{(3)}) T_3(PT_3)^{n-1} x_n] - x_n \| \\ &\le \| T_3(PT_3)^{n-1} x_n - x_n \| .\end{aligned}$$

that $\lim_{n\to+\infty} \| y_n - x_n \| = 0$. Also, we can see

$$\begin{aligned}\| z_n - y_n \| &= \| P[\alpha_n^{(2)} y_n + (1-\alpha_n^{(2)}) T_2(PT_2)^{n-1} y_n] - y_n \| \\ &\le \| [\alpha_n^{(2)} y_n + (1-\alpha_n^{(2)}) T_2(PT_2)^{n-1} y_n] - y_n \| \\ &\le \| T_2(PT_2)^{n-1} y_n - y_n \|\end{aligned}$$

and

$$\begin{aligned}\| x_{n+1} - z_n \| &= \| P[\alpha_n^{(1)} z_n + (1-\alpha_n^{(1)}) T_1(PT_1)^{n-1} z_n] - z_n \| \\ &\le \| [\alpha_n^{(1)} z_n + (1-\alpha_n^{(1)}) T_1(PT_1)^{n-1} z_n] - z_n \| \\ &\le \| T_1(PT_1)^{n-1} z_n - z_n \| .\end{aligned}$$

It follows from (3.5) and (3.6) that

$$\lim_{n\to+\infty} \| x_{n+1} - z_n \| = \lim_{n\to+\infty} \| z_n - y_n \| = 0.$$

Hence $\lim_{n\to+\infty} \| x_{n+1} - x_n \| = \lim_{n\to+\infty} \| z_n - x_n \| = 0$. Thus for any fixed $k \in N$,

$$\lim_{n\to+\infty} \| x_{n+k} - x_n \| = 0.$$

Noting (3.7) and

$$\begin{aligned}&\| x_n - T_3(PT_3)^{k-1} x_n \| \\ &\le \| x_n - x_{n+k} \| + \| x_{n+k} - T_3(PT_3)^{n+k-1} x_{n+k} \| + \| T_3(PT_3)^{n+k-1} x_{n+k} \\ &\quad - T_3(PT_3)^{n+k-1} x_n \| + \| T_3(PT_3)^{n+k-1} x_n - T_3(PT_3)^{k-1} x_n \| \\ &\le \| x_n - x_{n+k} \| + \| x_{n+k} - T_3(PT_3)^{n+k-1} x_{n+k} \| + \| x_{n+k} - x_n \| + r_{n+k} \\ &\quad + \| T_3(PT_3)^{n-1} x_n - x_n \| + r_k\end{aligned}$$

we have $\limsup_{n\to+\infty} \| x_n - T_3(PT_3)^{k-1} x_n \| \le r_k$, which implies

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \| x_n - T_3(PT_3)^{k-1} x_n \| = 0.$$

Combining (3.6) with

$$\begin{aligned}&\| T_2(PT_2)^{n-1} x_n - x_n \| \\ &\le \| T_2(PT_2)^{n-1} x_n - T_2(PT_2)^{n-1} y_n \| + \| T_2(PT_2)^{n-1} y_n - y_n \| + \| y_n - x_n \| \\ &\le 2 \| x_n - y_n \| + \| T_2(PT_2)^{n-1} y_n - y_n \| + r_n,\end{aligned}$$

we can see $\lim_{n\to+\infty} \| T_2(PT_2)^{n-1} x_n - x_n \| = 0$. Thus

$$\begin{aligned}&\| x_n - T_2(PT_2)^{k-1} x_n \| \\ &\le \| x_n - x_{n+k} \| + \| x_{n+k} - T_2(PT_2)^{n+k-1} x_{n+k} \| + \| T_2(PT_2)^{n+k-1} x_{n+k} \\ &\quad - T_2(PT_2)^{n+k-1} x_n \| + \| T_2(PT_2)^{n+k-1} x_n - T_2(PT_2)^{k-1} x_n \| \\ &\le \| x_n - x_{n+k} \| + \| x_{n+k} - T_2(PT_2)^{n+k-1} x_{n+k} \| + \| x_{n+k} - x_n \| + r_{n+k} \\ &\quad + \| T_2(PT_2)^{n-1} x_n - x_n \| + r_k\end{aligned}$$

which implies

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\| x_n - T_2(PT_2)^{k-1}x_n \| = 0.$$

Combining (3.5) with

$$\begin{aligned}&\| T_1(PT_1)^{n-1}x_n - x_n \| \\ \le& \| T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}z_n \| + \| T_1(PT_1)^{n-1}z_n - z_n \| + \| x_n - x_n \| \\ \le& 2 \| x_n - z_n \| + \| T_1(PT_1)^{n-1}z_n - z_n \| + r_n,\end{aligned}$$

we can see $\lim_{n\to+\infty}\| T_1(PT_1)^{n-1}x_n - x_n \| = 0$. Thus

$$\begin{aligned}&\| x_n - T_1(PT_1)^{k-1}x_n \| \\ \le& \| x_n - x_{n+k} \| + \| x_{n+k} - T_1(PT_1)^{n+k-1}x_{n+k} \| + \| T_1(PT_1)^{n+k-1}x_{n+k} \\ &-T_1(PT_1)^{n+k-1}x_n \| + \| T_1(PT_1)^{n+k-1}x_n - T_1(PT_1)^{k-1}x_n \| \\ \le& \| x_n - x_{n+k} \| + \| x_{n+k} - T_1(PT_1)^{n+k-1}x_{n+k} \| + \| x_{n+k} - x_n \| + r_{n+k} \\ &+ \| T_1(PT_1)^{n-1}x_n - x_n \| + r_k\end{aligned}$$

which implies

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\| x_n - T_1(PT_1)^{k-1}x_n \| = 0.$$

This completes the proof.

Define the operator $W_n : C \to C$ by

$$\begin{aligned}W_n x &= P[\alpha_n^{(1)}x^{(1)} + (1-\alpha_n^{(1)})T_1(PT_1)^{n-1}x^{(1)}];\\ x^{(1)} &= P[\alpha_n^{(2)}x^{(2)} + (1-\alpha_n^{(2)})T_2(PT_2)^{n-1}x^{(2)}];\\ x^{(2)} &= P[\alpha_n^{(3)}x + (1-\alpha_n^{(3)})T_3(PT_3)^{n-1}x],\end{aligned}$$

where $x \in C$. Then by (3.1), $x_{n+1} = W_n x_n$ and for all $x, y \in C$, we have

$$\begin{aligned}\| x^{(2)} - y^{(2)} \| &\le \alpha_n^{(3)} \| x - y \| + (1-\alpha_n^{(3)}) \| T_3(PT_3)^{n-1}x - T_3(PT_3)^{n-1}y \| \\ &\le \| x - y \| + r_n,\\ \| x^{(1)} - y^{(1)} \| &\le \| x^{(2)} - y^{(2)} \| + r_n \le \| x - y \| + 2r_n\end{aligned}$$

and

$$\| W_n x - W_n y \| \le \| x - y \| + 3r_n = \| x - y \| + w_n.$$

For any $f \in F$, we get $W_n f = f$. Set

$$S_{n,m} = W_{n+m-1}W_{n+m-2}\cdots W_{n+1}W_n : C \to C,$$

then $x_{n+m} = S_{n,m}x_n$ and for all $f \in F$, $S_{n,m}f = f$. Note that for any $x, y \in C$,

$$\| S_{n,m}x - S_{n,m}y \| \le \| x - y \| + (w_n + \cdots + w_{n+m-1}). \tag{3.8}$$

Lemma 3.3 *Let $f, g \in F$ and $\lambda \in [0, 1]$, then*

$$h(\lambda) = \lim_{n\to+\infty}\| \lambda x_n + (1-\lambda)f - g \|$$

exists.

Proof. It follows from Lemma 3.1 that $\lim_{n\to+\infty} \| x_n - f \| = r$ exists. If $\lambda = 0$, 1 or $r = 0$, then the conclusion holds. Assume that $r > 0$ and $\lambda \in (0, 1)$, then for any $\varepsilon > 0$, there exists $d > 0$ ($d < \varepsilon$) such that

$$(r + d)[1 - 2\lambda(1 - \lambda)\delta(\frac{\varepsilon}{r + d})] < r - d, \tag{3.9}$$

where $\delta(\cdot)$ is the modulus of convexity of the norm. Hence there exists a positive integer n_0 such that for all $n > n_0$,

$$r - \frac{d}{4} \le \| x_n - f \| \le r + \frac{d}{4} \tag{3.10}$$

and

$$\sum_{i=n}^{+\infty} w_i \le \lambda(1 - \lambda)\frac{d}{4} < \frac{\varepsilon}{4} \tag{3.11}$$

Now we claim that for all $n > n_0$,

$$\| S_{n,m}[\lambda x_n + (1 - \lambda)f] - [\lambda S_{n,m}x_n + (1 - \lambda)f] \| \le \varepsilon. \quad \forall m \in N$$

Otherwise, we can suppose that there are some $n > n_0$ and some $m \in N$ such that

$$\| S_{n,m}[\lambda x_n + (1 - \lambda)f] - [\lambda S_{n,m}x_n + (1 - \lambda)f] \| \ge \varepsilon.$$

Put $z = \lambda x_n + (1 - \lambda)f$, $x = (1 - \lambda)(S_{n,m}z - f)$, and $y = \lambda(S_{n,m}x_n - S_{n,m}z)$, then by (3.8), (3.10), and (3.11), we have

$$\begin{aligned}
\| x \| &= (1 - \lambda) \| S_{n,m}z - f \| \\
&\le (1 - \lambda)[\| z - f \| + (w_{n+m-1} + \cdots + w_{n+1} + w_n)] \\
&\le \lambda(1 - \lambda)(\| x_n - f \| + \frac{d}{4}) \\
&\le \lambda(1 - \lambda)(r + d), \\
\| y \| &= \lambda \| S_{n,m}x_n - S_{n,m}z \| \\
&\le \lambda[\| x_n - z \| + (w_{n+m-1} + \cdots + w_{n+1} + w_n)] \\
&\le \lambda(1 - \lambda)(\| x_n - f \| + \frac{d}{4}) \\
&\le \lambda(1 - \lambda)(r + d), \\
\| x - y \| &= \| S_{n,m}[\lambda x_n + (1 - \lambda)f] - [\lambda S_{n,m}x_n + (1 - \lambda)f] \| \ge \varepsilon
\end{aligned}$$

and

$$\lambda x + (1 - \lambda)y = \lambda(1 - \lambda)(S_{n,m}x_n - f).$$

So by Lemma 2.3, we get

$$\begin{aligned}
\lambda(1 - \lambda) \| S_{n,m}x_n - f \| &= \| \lambda x + (1 - \lambda)y \| \\
&\le \lambda(1 - \lambda)(r + d)[1 - 2\lambda(1 - \lambda)\delta(\frac{\varepsilon}{\lambda(1 - \lambda)(r + d)})] \\
&\le \lambda(1 - \lambda)(r + d)[1 - 2\lambda(1 - \lambda)\delta(\frac{\varepsilon}{r + d})]
\end{aligned}$$

and then by (3.10),

$$r - d \leq \| x_{n+m} - f \| = \| S_{n,m}x_n - f \| \\ \leq (r+d)[1 - 2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d})],$$

which contradicts (3.9). Thus we can conclude that for all $n > n_0$,

$$\| S_{n,m}[\lambda x_n + (1-\lambda)f] - [\lambda S_{n,m}x_n + (1-\lambda)f] \| \leq \varepsilon, \quad \forall m \in N.$$

Hence by (3.11), for all $n > n_0$,

$$\begin{aligned} & \| \lambda x_{n+m} + (1-\lambda)f - g \| \\ =& \| \lambda S_{n,m}x_n + (1-\lambda)f - g \| \\ \leq& \| [\lambda S_{n,m}x_n + (1-\lambda)f] - S_{n,m}[\lambda x_n + (1-\lambda)f] \| + \| S_{n,m}[\lambda x_n + (1-\lambda)f] - g \| \\ \leq& \varepsilon + \| \lambda x_n + (1-\lambda)f - g \| + (w_{n+m-1} + \cdots + w_{n+1} + w_n) \\ \leq& 2\varepsilon + \| \lambda x_n + (1-\lambda)f - g \| . \end{aligned}$$

For any fixed $n > n_0$, we can take the limsup for m and obtain

$$\limsup_{m\to+\infty} \| \lambda x_m + (1-\lambda)f - g \| \leq \| \lambda x_n + (1-\lambda)f - g \| + 2\varepsilon.$$

Hence

$$\limsup_{m\to+\infty} \| \lambda x_m + (1-\lambda)f - g \| \leq \liminf_{n\to+\infty} \| \lambda x_n + (1-\lambda)f - g \| + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$h(\lambda) = \lim_{n\to+\infty} \| \lambda x_n + (1-\lambda)f - g \|$$

exists. This completes the proof.

Remark 3.1 *If the mappings are asymptotically nonexpansive, we can use Bruck's Lemma* [23]*to prove Lemma 3.3. While Bruck's Lemma is not valid for non-Lipschitzian mappings, we must introduce new technique to establish a similar inequality. In* [20], *Chidume et al. also proved that* $\lim_{n\to+\infty} \| \lambda x_n + (1-\lambda)f - g \|$*exists (Lemma 3.12 in* [20]). *As we have seen, our proof is completely different from theirs in* [20].

Lemma 3.4 *If* $f \in \omega_\omega(\{x_n\})$ *and* $\lim_{n\to+\infty} \| \lambda x_n + (1-\lambda)f - g \|$*exists, then*

$$h(\lambda) = \lim_{n\to+\infty} \| \lambda x_n + (1-\lambda)f - g \| \leq \| f - g \| .$$

Proof. For any $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$\| \lambda x_n + (1-\lambda)f - g \| \leq h(\lambda) + \varepsilon.$$

Then for all $n \geq n_0$,

$$\langle \lambda x_n + (1-\lambda)f - g, J(f-g) \rangle \leq \| f - g \| (h(\lambda) + \varepsilon).$$

Since $f \in \omega_\omega(\{x_n\})$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ with $x_{n_i} \rightharpoonup f$. Hence $f \in \bar{co}\{x_{n_i}, i \geq n_0\}$ and

$$\{\lambda f + (1-\lambda)f - g, J(f-g)\} \leq \| f - g \| (h(\lambda) + \varepsilon),$$

i.e., $||f - g||^2 \leq ||f - g||(h(\lambda) + \varepsilon)$. Therefore $||f - g|| \leq h(\lambda)$. This completes the proof.

Now we can prove the weak convergence theorem of the iterative sequence (3.1).

Theorem 3.1 *Let C be a nonempty closed convex subset of uniformly convex Banach space X which satisfies the Opial's condition or whose dual X^* has the Kadec-Klee property. Let $P : X \to C$ be a nonexpansive retraction from X onto C. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive in the intermediate sense nonself mappings with $F \neq \emptyset$ and the nonnegative sequence $\{r_n\}$ satisfy $\sum_{n=1}^{+\infty} r_n < +\infty$. Let $\{x_n\}$ be defined by: $x_1 \in C$ and*

$$x_{n+1} = P[\alpha_n^{(1)} z_n + (1 - \alpha_n^{(1)}) T_1 (PT_1)^{n-1} z_n];$$
$$z_n = P[\alpha_n^{(2)} y_n + (1 - \alpha_n^{(2)}) T_2 (PT_2)^{n-1} y_n];$$
$$y_n = P[\alpha_n^{(3)} x_n + (1 - \alpha_n^{(3)}) T_3 (PT_3)^{n-1} x_n].$$

where $\{\alpha_n^{(i)}\}$is in [0, 1] with $0 < p \leq \alpha_n^{(i)} \leq q < 1$, $i = 1, 2, 3$. Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge weakly to a common fixed point of $\{T_i\}_{i=1}^3$.

Proof. It suffices to show that $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^3$. To this aim, we only need to prove that the set $\omega_\omega(\{x_n\})$ is singleton. Since X is reflexive and C is bounded, we obtain $\omega_\omega(\{x_n\}) \neq \emptyset$. Assume that $f, g \in \omega_\omega(\{x_n\})$, then there exist two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$in $\{x_n\}$ such that $x_{n_i} \rightharpoonup f$and $x_{n_j} \rightharpoonup g$. In the following, we shall show $f = g$. By Lemmas 2.5 and 3.2, $f, g \in F$. On one hand, if X satisfies the Opial's condition and $f \neq g$, then by the Lemma 3.1, we get

$$\begin{aligned} r = \lim_{n\to+\infty} \| x_n - f \| &= \lim_{i\to+\infty} \| x_{n_i} - f \| \\ < \lim_{i\to+\infty} \| x_{n_i} - g \| &= \lim_{n\to+\infty} \| x_n - g \| = \lim_{j\to+\infty} \| x_{n_j} - g \| \\ < \lim_{j\to+\infty} \| x_{n_j} - f \| &= \lim_{n\to+\infty} \| x_n - f \| = r. \end{aligned}$$

This contraction implies $f = g$. On the other hand, if X^* has Kadec-Klee property, then from Lemmas 2.4, 3.3, and 3.4, we have

$$\begin{aligned} &\| \lambda x_n + (1 - \lambda) f - g \|^2 \\ &\leq \| f - g \|^2 + 2\lambda \langle x_n - f, J(\lambda x_n + (1 - \lambda) f - g) \rangle \end{aligned}$$

and for all $\lambda \in [0, 1]$,

$$\liminf_{n\to+\infty} \langle x_n - f, J(\lambda x_n + (1 - \lambda) f - g) \rangle \geq 0.$$

Hence

$$\liminf_{j\to+\infty} \langle x_{n_j} - f, J(\lambda x_{n_j} + (1 - \lambda) f - g) \rangle \geq 0.$$

Thus for arbitrary $k \in N$, there exists $j_k \geq k$, $\{j_k\} \uparrow$, such that

$$\langle x_{n_{j_k}} - f, J(\frac{1}{k} x_{n_{j_k}} + (1 - \frac{1}{k}) f - g) \rangle \geq -\frac{1}{k}. \tag{3.12}$$

Obviously $x_{n_{j_k}} \rightharpoonup g$. Put

$$j_k = J(\frac{1}{k} x_{n_{j_k}} + (1 - \frac{1}{k}) f - g),$$

then we may assume that, without loss of generality, j_k is weakly convergent to some point $j \in X^*$. Therefore $\| j \| \leq \liminf_{k \to +\infty} \| j_k \| = \| f - g \|$. Noting

$$\langle f - g, j_k \rangle = \| \frac{1}{k} x_{n_{j_k}} + (1 - \frac{1}{k}) f - g \|^2 - \frac{1}{k} \langle x_{n_{j_k}} - f, j_k \rangle$$

and passing the limit for k, we have $\langle f - g, j \rangle = ||f - g||^2$. Hence $||j|| \geq ||f - g||$ and

$$\langle f - g, j \rangle = \| f - g \|^2 = \| j \|^2,$$

which means $j = J(f - g)$. Thus we can conclude $j_k \rightharpoonup j$ and $||j_k|| \to ||f - g|| = ||j||$. Since X^* has Kadec-Klee property, we have $j_k \to j$. Taking the limit in (3.12), we get $\langle g - f, j \rangle \geq 0$, i.e., $||f - g||^2 \leq 0$, which implies $f = g$. This completes the proof.

Remark 3.2 *Theorem 3.1 extends the main results in* [4,16,18,20]*to the case of asymptotically nonexpansive in the intermediate sense mappings and it seems to be new even in the case that the space has a Fréchet differentiable norm.*

In the following, we shall give a nonself mapping which is asymptotically nonexpansive in the intermediate sense but not asymptotically nonexpansive.

Example 3.1 Let Δ be the Cantor ternary set. Define the Cantor ternary function

$$\tau(x) = \begin{cases} \sum_{n=1}^{+\infty} \frac{b_n}{2^n} & x = \sum_{n=1}^{+\infty} \frac{2b_n}{3^n} \in \Delta, (b_n = 0, 1) \\ \sup\{\tau(y), y \leq x, y \in \Delta\} & x \in [0, 1] \backslash \Delta \end{cases}$$

then $\tau : [0, 1] \to [0, 1]$ is a continuous and increasing but not absolutely continuous function with $\tau(0) = 0$, $\tau(\frac{1}{2}) = \frac{1}{2}$ (see [27]). Since a Lipschitzian function is absolutely continuous, τ is non-Lipschitzian. Define ϕ: $R \to R$ by

$$\varphi(x) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \\ \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}\tau(1 - x) & \frac{1}{2} < x \leq 1 \end{cases}$$

It is easy to see that ϕ is continuous and for all $x, y \in R$, $|\varphi(x) - \varphi(y)| \leq \frac{1}{2}$. It also can be verified that the n-fold composition mapping ϕ^n is defined by

$$\varphi^n(x) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \\ \frac{x}{2^n} & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2^n}\tau(1 - x) & \frac{1}{2} < x \leq 1 \end{cases}$$

Since τ is non-Lipschitzian, so is ϕ^n and for all $x, y \in R$,

$$|\varphi^n(x) - \varphi^n(y)| \leq \frac{1}{2^n}.$$

Taking $X = R^2$ with the norm $\| (x, y) \| = \sqrt{x^2 + y^2}$, $(x, y) \in X$ and $C = R \times \{0\}$, we define the nonself mapping $T : C \to X$ by

$$T(x, 0) = (\varphi(x), x), \quad (x, 0) \in C,$$

then T is continuous and (0, 0) is a fixed point of T. Define $P : X \to C$ by

$$P(x, y) = (x, 0), \quad (x, y) \in X,$$

then P is a nonexpansive retraction from X onto C. Hence for all $(x, 0), (y, 0) \in C$,

$$T(PT)^{n-1}(x, 0) = (\varphi^n(x), \varphi^{n-1}(x)),$$

which means $T(PT)^{n\text{-}1}$ is is non-Lipschitzian and

$$\begin{aligned}
&\| T(PT)^{n-1}(x,0) - T(PT)^{n-1}(y,0) \| \\
&= \| (\varphi^n(x), \varphi^{n-1}(x)) - (\varphi^n(y), \varphi^{n-1}(y)) \| \\
&= \sqrt{(\varphi^n(x) - \varphi^n(y))^2 + (\varphi^{n-1}(x) - \varphi^{n-1}(y))^2} \\
&\leq |\varphi^n(x) - \varphi^n(y)| + |\varphi^{n-1}(x) - \varphi^{n-1}(y)| \\
&\leq \| (x,0) - (y,0) \| + \frac{3}{2^n}.
\end{aligned}$$

Therefore, we can conclude that T is asymptotically nonexpansive in the intermediate sense but not an asymptotically nonexpansive.

If T_1, T_2, and T_3 are nonexpansive, we can prove the following theorem.

Theorem 3.2 *Let C be a nonempty closed convex subset of uniformly convex Banach space X which satisfies the Opial's condition or whose dual has the Kadec-Klee property. Let* $P : X \to C$ *be a nonexpansive retraction from X onto C. Let* $T_1, T_2, T_3 : C \to X$ *be three nonexpansive nonself mappings and* $\{x_n\}$ *be defined by:* $x_1 \in C$ *and*

$$\begin{aligned}
x_{n+1} &= P[\alpha_n^{(1)} z_n + (1 - \alpha_n^{(1)}) T_1 z_n]; \\
z_n &= P[\alpha_n^{(2)} y_n + (1 - \alpha_n^{(2)}) T_2 y_n]; \\
y_n &= P[\alpha_n^{(3)} x_n + (1 - \alpha_n^{(3)}) T_3 x_n].
\end{aligned}$$

where $\{\alpha_n^{(i)}\}$*is in* $0[1]$*with* $0 < p \leq \alpha_n^{(i)} \leq q < 1$, $i = 1, 2, 3$. *Then* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge weakly to a common fixed point of* $\{T_i\}_{i=1}^3$.

Remark 3.3 *We would like to remark that if the so-called error terms are added in our recursion formula and are assumed to be bounded, then the results of this article still hold. Thus we can get the main results in* [19].

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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