Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 414232, 14 pages
doi:10.1155/2010/414232

Research Article

On the Fixed-Point Set of a Family of Relatively Nonexpansive and Generalized Nonexpansive Mappings

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Received 13 November 2009; Accepted 18 January 2010

Academic Editor: Tomonari Suzuki

We prove that the set of common fixed points of a given countable family of relatively nonexpansive mappings is identical to the fixed-point set of a single strongly relatively nonexpansive mapping. This answers Kohsaka and Takahashi's question in positive. We also introduce the concept of strongly generalized nonexpansive mappings and prove the analogue version of the result above for Ibaraki-Takahashi's generalized nonexpansive mappings. The duality theorem for two classes of strongly relatively nonexpansive mappings and of strongly generalized nonexpansive mappings is proved.

1. Introduction

Let C be a subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In this paper, the fixed-point set of the mapping T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. In 1973, Bruck [1] proved that for a given countable family of nonexpansive mappings in a strictly convex Banach space there exists a single nonexpansive mapping whose fixed-point set is identical to the set of common fixed points of the family. More precisely, the following is obtained.

Theorem 1.1. *Let C be a closed convex subset of a strictly convex Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is* *a sequence in* $(0,1)$ *such that* $\sum_{i=1}^{\infty}\alpha_i = 1$ *and* $U : C \to E$ *is defined by*

$$Ux = \sum_{i=1}^{\infty}\alpha_i T_i x \quad \text{for each } x \in C. \tag{1.1}$$

Then U *is nonexpansive and* $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$.

Recall that E is *strictly convex* if whenever x and y are norm-one elements in E satisfying $\|x + y\| = 2$ it follows that $x = y$. It is worth mentioning that Bruck's result above remains true for the class of quasi-nonexpansive mappings, that is, the set of common fixed points of a countable family of quasi-nonexpansive mappings is identical to the fixed-point set of a single quasi-nonexpansive mapping. A mapping $T: C \to E$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in F(T)$.

In 2004, Matsushita and Takahashi [2–4] introduced the so-called relatively nonexpansive mappings in Banach spaces. This class of mappings includes the resolvent of a maximal monotone operator and Alber's generalized projection. For more examples, we refer to [2–6]. Recently, Kohsaka and Takahashi [7] proved an analogue version of Bruck's result for a family of relatively nonexpansive mappings and they asked the following question.

Question 1. *For a given countable family of relatively nonexpansive mappings, is there a single strongly relatively nonexpansive mapping such that its fixed-point set is identical to the set of common fixed points of the family?*

A positive answer to this question is given in [7] for a finite family of mappings. The purpose of this paper is to give the answer of Kohsaka and Takahashi's question in positive. We also introduce a concept of strongly generalized nonexpansive mappings and present the analogue version of the result above for Ibaraki-Takahashi's generalized nonexpansive mappings. Finally, inspired by [8], we prove the duality theorem for two classes of strongly relatively nonexpansive mappings and of strongly generalized nonexpansive mappings.

2. Preliminaries

We collect together some definitions and preliminaries which are needed in this paper. The strong and weak convergences of a sequence $\{x_n\}$ in a Banach space E to an element $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A Banach space E is *uniformly convex* if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $\|x_n+y_n\| \to 2$ it follows that $x_n - y_n \to 0$. It is known that if E is uniformly convex, then it is reflexive and strictly convex. We say that E is *uniformly smooth* if the dual space E^* of E is uniformly convex. A Banach space E is *smooth* if the limit $\lim_{t\to 0}(\|x+ty\|-\|x\|)/t$ exists for all norm-one elements x and y in E. It is not hard to show that if E is reflexive, then E is smooth if and only if E^* is strictly convex. The value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^*\rangle$. The *duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\} \tag{2.1}$$

for all $x \in E$. The following facts are known (e.g., see [9, 10]).

(a) If E is smooth, then J is single valued.

(b) If E is strictly convex, then J is one-to-one, that is, $x \neq y$ implies that $Jx \cap Jy = \emptyset$.

(c) If E is reflexive, then J is onto.

(d) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

For a smooth Banach space E, Alber [5] considered the functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E. \tag{2.2}$$

Using this functional, Matsushita and Takahashi [2–4] studied and investigated the following mappings in Banach spaces. Suppose that C is a subset of a smooth Banach space E. A mapping $T : C \to E$ is *relatively nonexpansive* if the following properties are satisfied.

(R1) $F(T) \neq \emptyset$.

(R2) $\varphi(p, Tx) \leq \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$.

(R3) $I - T$ is demiclosed at zero, that is; whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

In a Hilbert space H, the duality mapping J is an identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $T : C \to H$ is relatively nonexpansive, then it is quasi-nonexpansive and $I - T$ is demiclosed at zero.

Recently, Kohsaka and Takahashi [7] proved an analogue version of Bruck's result for a family of relatively nonexpansive mappings. More precisely, they obtained the following.

Theorem 2.1 (see [7, Theorem 3.4]). *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E and let $\{T_i : C \to E\}_{i=1}^{m}$ be a finite family of relatively nonexpansive mappings such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{m} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{m} \subset (0,1)$ are finite sequences such that $\sum_{i=1}^{m} \alpha_i = 1$ and $R : C \to E$ is defined by*

$$Rx = J^{-1}\left(\sum_{i=1}^{m} \alpha_i(\beta_i Jx + (1 - \beta_i) JT_i x)\right) \quad \text{for each } x \in C. \tag{2.3}$$

Then R is strongly relatively nonexpansive and $F(R) = \bigcap_{i=1}^{m} F(T_i)$.

Recall that a relatively nonexpansive mapping $T : C \to E$ [6] is *strongly relatively nonexpansive* if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p, x_n) - \varphi(p, Tx_n) \to 0$ for some $p \in F(T)$ it follows that $\varphi(Tx_n, x_n) \to 0$.

To obtain the result for a countable family of relatively nonexpansive mappings, the same authors proved the following result.

Theorem 2.2 (see [7, Theorem 3.3]). *Let C and E be as in Theorem 2.1 and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $U : C \to E$ is defined by*

$$Ux = J^{-1}\left(\sum_{i=1}^{\infty}\alpha_i(\beta_i Jx + (1-\beta_i)JT_i x)\right) \quad \text{for each } x \in C. \tag{2.4}$$

Then U is relatively nonexpansive and $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$.

Remark 2.3. They also asked the question of whether the mapping U in Theorem 2.2 is strongly relatively nonexpansive (see [7, Problem 3.5]).

The following lemmas are needed in proving the result.

Lemma 2.4 (see [11, Theorem 2]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|) \tag{2.5}$$

for all $\alpha \in [0,1]$ and $x, y \in B_r := \{z \in E : \|z\| \leq r\}$.

Lemma 2.5. *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\left\|\sum_{i=1}^{\infty}\alpha_i x_i\right\|^2 \leq \sum_{i=1}^{\infty}\alpha_i\|x_i\|^2 - \alpha_1\alpha_k g(\|x_1 - x_k\|), \tag{2.6}$$

for all $\{x_i\}_{i=1}^{\infty} \subset B_r$, $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ with $\sum_{i=1}^{\infty} \alpha_i = 1$, and $k \in \mathbb{N}$.

Proof. We note that both series $\sum_{i=1}^{\infty} \alpha_i x_i$ and $\sum_{i=1}^{\infty} \alpha_i \|x_i\|^2$ converge. For $r > 0$, let $g : [0, 2r] \to [0, \infty)$ be a function satisfying the properties of Lemma 2.4. Using the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\left\|\sum_{i=1}^{\infty}\alpha_i x_i\right\|^2 &\leq (\alpha_1 + \alpha_k)\left\|\frac{\alpha_1}{\alpha_1+\alpha_k}x_1 + \frac{\alpha_k}{\alpha_1+\alpha_k}x_k\right\|^2 + \sum_{i \neq 1,k}\alpha_i\|x_i\|^2 \\
&\leq (\alpha_1 + \alpha_k)\left(\frac{\alpha_1}{\alpha_1+\alpha_k}\|x_1\|^2 + \frac{\alpha_k}{\alpha_1+\alpha_k}\|x_k\|^2 - \frac{\alpha_1\alpha_k}{(\alpha_1+\alpha_k)^2}g(\|x_1 - x_k\|)\right) \\
&\quad + \sum_{i \neq 1,k}\alpha_i\|x_i\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}\alpha_i\|x_i\|^2 - \frac{\alpha_1\alpha_k}{(\alpha_1+\alpha_k)}g(\|x_1-x_k\|) \\
&\leq \sum_{i=1}^{\infty}\alpha_i\|x_i\|^2 - \alpha_1\alpha_k g(\|x_1-x_k\|).
\end{aligned}
\tag{2.7}
$$

This completes the proof. □

Lemma 2.6 (see [12, Lemma 2.10]). *Let E be a strictly convex Banach space and let $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ with $\sum_{i=1}^{\infty}\alpha_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in E such that both series $\sum_{i=1}^{\infty}\alpha_i x_i$ and $\sum_{i=1}^{\infty}\alpha_i\|x_i\|^2$ converge, and*

$$
\left\|\sum_{i=1}^{\infty}\alpha_i x_i\right\|^2 = \sum_{i=1}^{\infty}\alpha_i\|x_i\|^2, \tag{2.8}
$$

then $\{x_i\}_{i=1}^{\infty}$ is a constant sequence.

Lemma 2.7 (see [13, Proposition 2]). *Let E be a smooth and uniformly convex Banach space. Suppose that either $\{x_n\}$ or $\{y_n\}$ is a bounded sequence in E and $\varphi(x_n, y_n) \to 0$. Then $x_n - y_n \to 0$.*

3. Relatively Nonexpansive Mappings and Quasi-Nonexpansive Mappings

We first start with some observation which is a tool for proving Theorem 3.2.

Theorem 3.1. *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of mappings such that $\bigcap_{i=1}^{\infty}F(T_i) \neq \emptyset$ and*

$$
\varphi(p, T_i x) \leq \varphi(p, x) \quad \forall x \in C,\ p \in \bigcap_{i=1}^{\infty}F(T_i),\ i \in \mathbb{N}. \tag{3.1}
$$

Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\alpha_i = 1$ and $S : C \to E$ is defined by

$$
Sx = J^{-1}\left(\sum_{i=1}^{\infty}\alpha_i J T_i x\right) \quad \text{for each } x \in C. \tag{3.2}
$$

Let $\{x_n\}$ be a bounded sequence in C. Then the following are equivalent.

(a) $x_n - Sx_n \to 0$.

(b) $x_n - T_i x_n \to 0$ *for each* $i \in \mathbb{N}$.

In particular, $F(S) = \bigcap_{i=1}^{\infty}F(T_i)$.

Proof. For fixed $p \in \bigcap_{i=1}^{\infty} F(T_i)$ and $x \in C$, we have

$$(\|p\| - \|T_i x\|)^2 \leq \varphi(p, T_i x) \leq \varphi(p, x) \leq (\|p\| + \|x\|)^2. \tag{3.3}$$

In particular, $\|T_i x\| \leq \|x\| + 2\|p\|$ for all $i \in \mathbb{N}$ and $x \in C$. Hence, for each $x \in C$, the series $\sum_{i=1}^{\infty} \alpha_i J T_i x$ converges (absolutely). This implies that the mapping S is well defined.

Let $\{x_n\}$ be a bounded sequence in C. Suppose that

$$x_n - Sx_n \longrightarrow 0. \tag{3.4}$$

By the boundedness of $\{x_n\}$, we put

$$M := \sup\{\|x_n\| : n \in \mathbb{N}\} + 2\|p\| < \infty. \tag{3.5}$$

Then $\|JT_i x_n\| = \|T_i x_n\| \leq M$ for all $i, n \in \mathbb{N}$. We now consider the following estimates for each $k \in \mathbb{N}$ such that $k \neq 1$ and for any $n \in \mathbb{N}$:

$$\begin{aligned}
\varphi(p, Sx_n) &= \varphi\left(p, \sum_{i=1}^{\infty} \alpha_i J T_i x_n\right) = \|p\|^2 - 2\left\langle q, \sum_{i=1}^{\infty} \alpha_i J T_i x_n\right\rangle + \left\|\sum_{i=1}^{\infty} \alpha_i J T_i x_n\right\|^2 \\
&\leq \|p\|^2 - \sum_{i=1}^{\infty} 2\alpha_i \langle q, JT_i x_n\rangle + \sum_{i=1}^{\infty} \alpha_i \|JT_i x_n\|^2 - \alpha_1 \alpha_k g(\|JT_1 x_n - JT_k x_n\|) \\
&= \sum_{i=1}^{\infty} \alpha_i \varphi(p, T_i x_n) - \alpha_1 \alpha_k g(\|JT_1 x_n - JT_k x_n\|) \\
&\leq \varphi(p, x_n) - \alpha_1 \alpha_k g(\|JT_1 x_n - JT_k x_n\|),
\end{aligned} \tag{3.6}$$

where g is the function given in Lemma 2.5 associated with the uniform convexity of E^* and the number M. Notice that $\varphi(p, x_n) - \varphi(p, Sx_n) \to 0$. Consequently, for $k \neq 1$,

$$\alpha_1 \alpha_k g(\|JT_1 x_n - JT_k x_n\|) \longrightarrow 0. \tag{3.7}$$

This implies that

$$JT_1 x_n - JT_k x_n \longrightarrow 0. \tag{3.8}$$

We next prove that

$$JT_1 x_n - JSx_n \longrightarrow 0. \tag{3.9}$$

Let $\varepsilon > 0$ be given. We choose an integer K such that $\sum_{i=K+1}^{\infty} \alpha_i < (\varepsilon/4M)$. Since $JT_1 x_n - JT_i x_n \to 0$ as $n \to \infty$ for all $i = 1, \ldots, K$, we now choose an integer N such that

$$\|JT_1 x_n - JT_i x_n\| < \frac{\varepsilon}{2} \tag{3.10}$$

for all $n \geq N$ and $i = 2, \ldots, K$. Then, if $n \geq N$,

$$\begin{aligned}\|JT_1x_n - JSx_n\| &= \left\|\sum_{i=2}^{\infty}\alpha_i(JT_1x_n - JT_ix_n)\right\| \\ &\leq \sum_{i=2}^{K}\alpha_i\|JT_1x_n - JT_ix_n\| + \sum_{i=K+1}^{\infty}\alpha_i\|JT_1x_n - JT_ix_n\| \\ &< \left(\sum_{i=2}^{K}\alpha_i\right)\frac{\varepsilon}{2} + \left(\sum_{i=K+1}^{\infty}\alpha_i\right)2M < \varepsilon.\end{aligned} \tag{3.11}$$

This implies that (3.9) holds. In particular, since J^{-1} is uniformly norm-to-norm continuous on each bounded set, we can conclude from (3.8) that

$$T_1x_n - T_kx_n \longrightarrow 0 \quad \text{for each } k \neq 1 \tag{3.12}$$

and from (3.9) that

$$T_1x_n - Sx_n \longrightarrow 0. \tag{3.13}$$

This together with (3.4) gives

$$T_1x_n - x_n \longrightarrow 0. \tag{3.14}$$

Assertion (b) follows immediately from (3.12) and (3.14).

Conversely, we assume that $x_n - T_ix_n \rightarrow 0$ for each $i \in \mathbb{N}$. Since J is uniformly norm-to-norm continuous on each bounded set,

$$Jx_n - JT_ix_n \longrightarrow 0 \quad \text{for each } i \in \mathbb{N}. \tag{3.15}$$

We show that

$$Jx_n - JSx_n \longrightarrow 0. \tag{3.16}$$

Let $\varepsilon > 0$. Then there exist positive integers K, N such that $\sum_{i=K+1}^{\infty}\alpha_i < (\varepsilon/4M)$ and

$$\|Jx_n - JT_ix_n\| < \frac{\varepsilon}{2} \tag{3.17}$$

for all $n \geq N$ and $i = 1, \dots, K$. If $n \geq N$, then

$$\begin{aligned} \|Jx_n - JSx_n\| &= \left\| \sum_{i=1}^{\infty} \alpha_i (Jx_n - JT_i x_n) \right\| \\ &\leq \sum_{i=1}^{K} \alpha_i \|Jx_n - JT_i x_n\| + \sum_{i=K+1}^{\infty} \alpha_i \|Jx_n - JT_i x_n\| \\ &< \left(\sum_{i=1}^{K} \alpha_i \right) \frac{\varepsilon}{2} + \left(\sum_{i=K+1}^{\infty} \alpha_i \right) 2M < \varepsilon. \end{aligned} \tag{3.18}$$

By the uniform norm-to-norm continuity of J^{-1} on each bounded set, we can conclude assertion (a) from (3.16). This completes the proof. □

Theorem 3.2. *Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $S : C \to E$ is defined by*

$$Sx = J^{-1} \left(\sum_{i=1}^{\infty} \alpha_i JT_i x \right) \quad \text{for each } x \in C. \tag{3.19}$$

Then S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. To show that S is relatively nonexpansive, we prove only that $I - S$ is demiclosed at zero. Suppose that $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup p \in C$ and $x_n - Sx_n \to 0$. From Theorem 3.1, we have $x_n - T_i x_n \to 0$ for each $i \in \mathbb{N}$. Since each $I - T_i$ is demiclosed at zero, $p \in F(T_i)$. Consequently, $p \in \bigcap_{i=1}^{\infty} F(T_i) = F(S)$, as desired. □

We now give an answer of Kohsaka and Takahashi's question in positive.

Theorem 3.3. *The mapping U in Theorem 2.2 is strongly relatively nonexpansive.*

Proof. The mapping U can be rewritten as

$$\begin{aligned} U &= J^{-1} \left(\sum_{i=1}^{\infty} \alpha_i (\beta_i J + (1 - \beta_i) JT_i) \right) \\ &= J^{-1} \left(\left(\sum_{i=1}^{\infty} \alpha_i \beta_i \right) J + \sum_{i=1}^{\infty} \alpha_i (1 - \beta_i) JT_i \right) = J^{-1} \left(\sum_{i=0}^{\infty} \gamma_i JT_i \right), \end{aligned} \tag{3.20}$$

where T_0 is the identity mapping, $\gamma_0 = \sum_{i=1}^{\infty} \alpha_i \beta_i > 0, \gamma_i = \alpha_i (1 - \beta_i) > 0$ for all $i \in \mathbb{N}$, and $\sum_{i=0}^{\infty} \gamma_i = 1$. It follows from Theorem 3.2 that $S := J^{-1}(\sum_{i=1}^{\infty} \widehat{\gamma}_i JT_i)$ is relatively nonexpansive, where $\widehat{\gamma}_i \equiv \gamma_i / (\sum_{i=1}^{\infty} \gamma_i)$. Consequently, by Theorem 2.1 with $m = 1$, the mapping

$$U = J^{-1}(\gamma_0 J + (1 - \gamma_0) JS) \tag{3.21}$$

is strongly relatively nonexpansive. □

Using the same idea as in Theorem 3.1, we also have the following result whose proof is left to the reader to verify.

Theorem 3.4. *Let C be a closed convex subset of a uniformly convex Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of quasi-nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $U : C \to E$ is defined by*

$$Ux = \sum_{i=1}^{\infty} \alpha_i T_i x \quad \text{for each } x \in C. \tag{3.22}$$

Then $I - U$ is demi-closed at zero if and only if each mapping $I - T_i$ is demi-closed at zero.

4. Ibaraki-Takahashi's Generalized Nonexpansive Mappings

Let C be a subset of a smooth Banach space E. In 2007, Ibaraki and Takahashi [14] introduced the following mapping. A mapping $T : C \to E$ is *generalized nonexpansive* if the following properties are satisfied:

(G1) $F(T) \neq \emptyset$,

(G2) $\varphi(Tx, p) \leq \varphi(x, p)$ for all $p \in F(T)$ and $x \in C$.

A mapping $T : C \to E$ satisfies *property (G3)* if whenever $\{x_n\}$ is a sequence in C such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $Jx_n - JTx_n \to 0$ it follows that $p \in F(T)$. Here $\overset{*}{\rightharpoonup}$ denotes the weak* convergence in the dual space.

The generalized resolvent $(I + \lambda BJ)^{-1}$ of the maximal monotone operator $B \subset E^* \times E$, where E is a smooth and uniformly convex Banach space, and the sunny generalized nonexpansive retraction from a strictly convex, smooth, and reflexive Banach space onto its closed subset are examples of generalized nonexpansive mappings satisfying property (G3) (see [15]). The relation between two classes of relatively nonexpansive mappings and of generalized nonexpansive mappings is recently obtained in [8].

The property (G3) of the mapping T and the demiclosedness of $I - T$ are related as shown in the following remark.

Remark 4.1. Let C be a subset of a smooth Banach space E and $T : C \to E$. Then the following assertions hold true.

(1) If E is uniformly smooth, the duality mapping J is weakly sequentially continuous, and T satisfies property (G3), then $I - T$ is demiclosed at zero.

(2) If E is uniformly convex, J^{-1} is weakly sequentially continuous, and $I - T$ is demiclosed at zero, then T satisfies property (G3).

Theorem 4.2. *Let C be a closed convex subset of a smooth Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of generalized nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\widehat{S} : C \to E$ is defined by*

$$\widehat{S}x = \sum_{i=1}^{\infty} \alpha_i T_i x \quad \text{for each } x \in C. \tag{4.1}$$

Then the mapping $\widehat{S}$ is well defined and the following assertions hold true.

(i) *If E is strictly convex, then $F(\widehat{S}) = \bigcap_{i=1}^{\infty} F(T_i)$ and $\widehat{S}$ is generalized nonexpansive.*

(ii) *If E is uniformly convex and $\{x_n\}$ is a bounded sequence in C, then the following statements are equivalent:*

(a) $x_n - \widehat{S}x_n \to 0$,

(b) $x_n - T_i x_n \to 0$ *for each* $i \in \mathbb{N}$.

(iii) *The mapping $I - \widehat{S}$ is demi-closed at zero if and only if each mapping $I - T_i$ is demi-closed at zero.*

(iv) *Suppose that E is uniformly convex and uniformly smooth. Then the mapping $\widehat{S}$ satisfies property $(G3)$ if and only if each mapping T_i satisfies property $(G3)$.*

Proof. Using some basic properties of the functional φ, we have $\|T_i x\| \leq \|x\| + 2\|p\|$ for all $x \in C$, $p \in F(T_i)$. Since $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, the sequence $\{T_i x\}_{i=1}^{\infty}$ is bounded for each $x \in C$ and, hence the series $\sum_{i=1}^{\infty} \alpha_i T_i x$ converges (absolutely). This implies that $\widehat{S}$ is well defined. For fixed $p \in \bigcap_{i=1}^{\infty} F(T_i)$ and $x \in C$, we have the following expressions:

$$\varphi\left(\widehat{S}x, p\right) = \varphi\left(\sum_{i=1}^{\infty} \alpha_i T_i x, p\right) = \left\|\sum_{i=1}^{\infty} \alpha_i T_i x\right\|^2 - 2\left\langle \sum_{i=1}^{\infty} \alpha_i T_i x, Jp \right\rangle + \|p\|^2,$$
$$\sum_{i=1}^{\infty} \alpha_i \|T_i x\|^2 - 2\left\langle \sum_{i=1}^{\infty} \alpha_i T_i x, Jp \right\rangle + \|p\|^2 = \sum_{i=1}^{\infty} \alpha_i \varphi(T_i x, p) \leq \varphi(x, p). \tag{4.2}$$

(i) The inclusion $\bigcap_{i=1}^{\infty} F(T_i) \subset F(\widehat{S})$ is obvious. To see the reverse inclusion, let $x \in F(\widehat{S})$. By the convexity of $\|\cdot\|^2$, $\varphi(\widehat{S}x, p) = \varphi(x, p)$, and the expressions of (4.2), we have

$$\left\|\sum_{i=1}^{\infty} \alpha_i T_i x\right\|^2 = \sum_{i=1}^{\infty} \alpha_i \|T_i x\|^2. \tag{4.3}$$

It follows from Lemma 2.6 that $\{T_i x\}_{i=1}^{\infty}$ is a constant sequence, and hence $x = \widehat{S}x = \sum_{i=1}^{\infty} \alpha_i T_i x = T_j x$ for all $j \in \mathbb{N}$. This implies that $x \in \bigcap_{i=1}^{\infty} F(T_i)$, that is, $F(\widehat{S}) \subset \bigcap_{i=1}^{\infty} F(T_i)$. Now $F(\widehat{S}) = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Again, using the convexity of $\|\cdot\|^2$, we can show that $\widehat{S}$ satisfies property (G2), and hence it is generalized nonexpansive, as desired.

(ii) Since the proof of this assertion is very similar to that of Theorem 3.1, it is omitted.

(iii) and (iv) follow directly from (ii). □

Remark 4.3. Theorem 4.2(i) generalizes [16, Theorem 3.3] from a finite family to a countable one.

Following Reich [6], we introduced the following concept. A generalized nonexpansive mapping $T : C \to E$ is *strongly generalized nonexpansive* if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(x_n, p) - \varphi(Tx_n, p) \to 0$ for some $p \in F(T)$ it follows that $\varphi(x_n, Tx_n) \to 0$.

Lemma 4.4. *Let C be a closed convex subset of a strictly convex and smooth Banach space E. Suppose that $T, S : C \to E$ is a generalized nonexpansive mapping and a strongly generalized nonexpansive mapping, respectively, and suppose that $F(T) \cap F(S) \neq \emptyset$. For $\alpha \in (0,1)$, let the mapping $U : C \to E$ be defined by*

$$Ux = \alpha Sx + (1-\alpha)Tx \quad \forall x \in C. \tag{4.4}$$

Then $F(U) = F(T) \cap F(S)$. If, in addition, E is uniformly convex, then U is strongly generalized nonexpansive.

Proof. The first assertion follows from Theorem 4.2(i). We now assume that E is uniformly convex. Suppose that $\{x_n\}$ is a bounded sequence in C such that $\varphi(x_n, p) - \varphi(Ux_n, p) \to 0$ for some $p \in F(U) = F(T) \cap F(S)$. It is clear that the sequences $\{Sx_n\}$ and $\{Tx_n\}$ are both bounded. By the uniform convexity of E, we have

$$\|\alpha Sx_n + (1-\alpha)Tx_n\|^2 \leq \alpha\|Sx_n\|^2 + (1-\alpha)\|Tx_n\|^2 - \alpha(1-\alpha)g(\|Sx_n - Tx_n\|), \tag{4.5}$$

where g is a function given by Lemma 2.4. Since T and S are generalized nonexpansive,

$$\begin{aligned} \varphi(Ux_n, p) &\leq \alpha\varphi(Sx_n, p) + (1-\alpha)\varphi(Tx_n, p) - \alpha(1-\alpha)g(\|Sx_n - Tx_n\|) \\ &\leq \varphi(x_n, p) - \alpha(1-\alpha)g(\|Sx_n - Tx_n\|). \end{aligned} \tag{4.6}$$

Consequently, $Sx_n - Tx_n \to 0$, and hence $Sx_n - Ux_n \to 0$. This implies that $\varphi(x_n, p) - \varphi(Sx_n, p) \to 0$. Since S is strongly generalized nonexpansive, $\varphi(x_n, Sx_n) \to 0$. It follows from Lemma 2.7 that $x_n - Sx_n \to 0$, and hence $x_n - Ux_n \to 0$. This implies that $\varphi(x_n, Ux_n) \to 0$ and U is strongly generalized nonexpansive, as desired. □

The following is an analogue version of Kohsaka and Takahashi's question for a countable family of generalized nonexpansive mappings.

Theorem 4.5. *Let C be a closed convex subset of a smooth and uniformly convex Banach space E and let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a countable family of generalized nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Then there exists a strongly generalized nonexpansive mapping $S : C \to E$ such that $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. Suppose that $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\alpha \in (0,1)$. We define $S : C \to E$ by

$$Sx = \alpha x + (1-\alpha)\sum_{i=1}^{\infty} \alpha_i T_i x \quad \forall x \in C. \tag{4.7}$$

Notice that $\sum_{i=1}^{\infty} \alpha_i T_i$ is generalized nonexpansive and $F(\sum_{i=1}^{\infty} \alpha_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$ by Theorem 4.2(i). Moreover, by Lemma 4.4 and the fact that the identity is strongly generalized nonexpansive, the conclusion is satisfied by the mapping S. □

5. Duality between Strongly Relatively Nonexpansive Mappings and Strongly Generalized Nonexpansive Mappings

Let C be a subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to E$ be a mapping. We can define the *duality* $T^* : JC \to E^*$ of T by (see [8])

$$T^* x^* = JTJ^{-1} x^* \quad \forall x^* \in JC. \tag{5.1}$$

We now consider a functional from $E^* \times E^*$ into $[0, \infty)$, still denoted by φ, by

$$\varphi(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^* y^* \rangle + \|y^*\|^2 \quad (x^*, y^* \in E^*), \tag{5.2}$$

where J^* is the duality mapping from E^* onto $E^{**} = E$. It is clear that $J^* = J^{-1}$. Then, whenever x, y are elements in E and x^*, y^* are elements in E^* satisfying $x^* = Jx$ and $y^* = Jy$, it follows that

$$\varphi(x^*, y^*) = \varphi(y, x). \tag{5.3}$$

Remark 5.1. The following assertions hold (see [8]).

(A) If $x \in C$ and $x^* = Jx$, then $T^* x^* = JTx$. In particular, $F(T^*) = JF(T)$. Moreover, if $\{x_n\}$ is a sequence in C and $x_n^* \equiv Jx_n$, then

(i) $x_n^* - T^* x_n^* \to 0$ if and only if $Jx_n - JTx_n \to 0$,

(ii) $J^* x_n^* - J^* T^* x_n^* \to 0$ if and only if $x_n - Tx_n \to 0$.

(B) If $x \in C, p \in F(T), x^* = Jx$ and $p^* = Jp$, then

$$\varphi(Jp, T^* x^*) = \varphi(Tx, p), \qquad \varphi(T^* x^*, Jp) = \varphi(p, Tx). \tag{5.4}$$

The following duality theorem is proved in [8].

Theorem 5.2. *Let C be a subset of a smooth, strictly convex and reflexive Banach space E and let $T : C \to E$ be a mapping. Suppose that $T^* : JC \to E^*$ is the duality of T. Then the following assertions hold true.*

(1) *If T is relatively nonexpansive, then T^* is generalized nonexpansive with property* $(G3)$.

(2) *If T is generalized nonexpansive with property* $(G3)$, *then T^* is relatively nonexpansive.*

We now prove the duality theorem for strongly relatively nonexpansive mappings and strongly generalized nonexpansive mappings.

Theorem 5.3. *Let C be a subset of a smooth, strictly convex and reflexive Banach space E and let T: $C \rightarrow E$ be a mapping. Suppose that T^*: $JC \rightarrow E^*$ is the duality of T. Then the following assertions hold true.*

(1) *If T is strongly relatively nonexpansive, then T^* is strongly generalized nonexpansive with property $(G3)$.*

(2) *If T is strongly generalized nonexpansive with property $(G3)$, then T^* is strongly relatively nonexpansive.*

Proof. We prove only (1) and leave (2) for the reader to verify. Suppose that $\{x_n^*\}$ is a bounded sequence in JC such that $\varphi(x_n^*, p^*) - \varphi(T^* x_n^*, p^*) \rightarrow 0$ for some $p^* \in F(T^*)$. We assume that $\{x_n\}$ is a sequence in C such that $Jx_n \equiv x_n^*$ and p is a point in $F(T)$ such that $Jp = p^*$. Clearly, $\{x_n\}$ is bounded. Moreover, by Remark 5.1, we have $\varphi(p, x_n) \equiv \varphi(x_n^*, p^*)$ and $\varphi(p, Tx_n) \equiv \varphi(T^* x_n^*, p^*)$. Consequently, $\varphi(p, x_n) - \varphi(p, Tx_n) \rightarrow 0$. It follows from the strongly relative nonexpansiveness that $\varphi(x_n^*, T^* x_n^*) = \varphi(Tx_n, x_n) \rightarrow 0$. This completes the proof. □

Acknowledgment

The corresponding author was supported by the Centre of Excellence in Mathematics, the Commission on Higher Education of Thailand.

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