Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 408086, 17 pages
doi:10.1155/2010/408086

Research Article

Common Fixed Point Theorem for Non-Self Mappings Satisfying Generalized Ćirić Type Contraction Condition in Cone Metric Space

R. Sumitra,[1] V. Rhymend Uthariaraj,[2] R. Hemavathy,[3] and P. Vijayaraju[4]

[1] *Department of Mathematics, SMK Fomra Institute of Technology, Chennai-603 103, Tamilnadu, India*
[2] *Ramanujan Computing Centre, Anna University Chennai, Chennai-600 025, Tamilnadu, India*
[3] *Department of Mathematics, Easwari Engineering College, Ramapuram, Chennai-600 089, Tamilnadu, India*
[4] *Department of Mathematics, Anna University Chennai, Chennai-600 025, Tamilnadu, India*

Correspondence should be addressed to R. Hemavathy, hemaths@rediffmail.com

Received 27 November 2009; Revised 21 February 2010; Accepted 9 March 2010

Academic Editor: Naseer Shahzad

We prove common fixed point theorem for coincidentally commuting nonself mappings satisfying generalized contraction condition of Ćirić type in cone metric space. Our results generalize and extend all the recent results related to non-self mappings in the setting of cone metric space.

1. Introduction

Recently, Huang and Zhang [1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The category of cone metric spaces is larger than metric spaces and there are different types of cones. Subsequently, many authors like Abbas and Jungck [2], Abbas and Rhoades [3], Ilić and Rakočević [4], Raja and Vaezpour [5] have generalized the results of Huang and Zhang [1] and studied the existence of common fixed points of a pair of self mappings satisfying a contractive type condition in the framework of normal cone metric spaces. However, authors like Janković et al. [6], Jungck et al. [7], Kadelburg et al. [8, 9], Radenović and Rhoades [10], Rezapour and Hamlbarani [11] studied the existence of common fixed points of a pair of self and nonself mappings satisfying a contractive type condition in the situation in which the cone does not need be normal.

The study of fixed point theorems for nonself mappings in metrically convex metric spaces was initiated by Assad and Kirk [12]. Utilizing the induction method of Assad and Kirk [12], many authors like Assad [13], Ćirić [14], Hadžić [15], Hadžić and Gajić [16], Imdad and Kumar [17], Rhoades [18, 19] have obtained common fixed point in metrically convex spaces. Recently, Ćirić and Ume [20] defined a wide class of multivalued nonself mappings which satisfy a generalized contraction condition and proved a fixed point theorem which generalize the results of Itoh [21] and Khan [22].

Very recently, Radenović and Rhoades [10] extended the fixed point theorem of Imdad and Kumar [17] for a pair of nonself mappings to nonnormal cone metric spaces. Janković et al. [6] proved new common fixed point results for a pair of nonself mappings defined on a closed subset of metrically convex cone metric space which is not necessarily normal by adapting Assad-Kirk's method.

The aim of this paper is to prove common fixed point theorems for coincidentally commuting nonself mappings satisfying a generalized contraction condition of Ćirić type in the setting of cone metric spaces. Our results generalize mainly results of Ćirić and Ume [20] and all the recent results related to nonself mappings in the setting of cone metric space.

2. Definitions and Preliminaries

We recall some basic definitions and preliminaries that will be needed in the sequel.

Definition 2.1 (see [1]). Let E be a real Banach space. A subset P of E is called a Cone if and only if

(1) P is nonempty, closed and $P \neq \{0\}$;

(2) $\alpha, \beta \in R$, $\alpha, \beta \geq 0$ and $x, y \in P \Rightarrow \alpha x + \beta y \in P$;

(3) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, a partial ordering is defined as $\leq$ on E with respect to P by $x \leq y$, if and only if $y - x \in P$. It is denoted as $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone $P \subset E$ is called normal, if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\|x\| \leq K\|y\|. \tag{2.1}$$

The least positive number K satisfying (2.1) is called the normal constant of P. It is clear that $K \geq 1$. There are nonnormal cones also.

The definition of a cone metric space given by Huang and Zhang [1] is as follows.

Definition 2.2 (see [1]). Let X be a nonempty set. Suppose that E is a real Banach space, P is a cone with $\operatorname{int} P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

If the mapping $d : X \times X \to E$ satisfies the following:

(1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x,y,z \in X$;

then d is called a cone metric on X and (X,d) is called a cone metric space.

Example 2.3 (see [1]). Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.4 (see [1]). Let (X,d) be a cone metric space and $\{x_n\}$ a sequence in X. Then, one has the following.

(1) $\{x_n\}$ converges to $x \in X$, if for every $c \in E$ with $0 \ll c$, there is $n_0 \in N$ such that for all $n \geq n_0$,

$$d(x_n, x) \ll c. \tag{2.2}$$

It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $(n \to \infty)$.

(2) If for any $c \in E$, there is a number $n_0 \in N$ such that for all $m,n \geq n_0$

$$d(x_n, x_m) \ll c, \tag{2.3}$$

then $\{x_n\}$ is called a Cauchy sequence in X.

(3) (X,d) is a complete cone metric space, if every Cauchy sequence in X is convergent.

(4) A self mapping $T : X \to X$ is said to be continuous at a point $x \in X$, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X.

The following two lemmas of Huang and Zhang [1] will be required in the sequel.

Lemma 2.5 (see [1]). *Let (X,d) be a cone metric space and P a normal cone with normal constant K. A sequence $\{x_n\}$ in X converges to x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.*

Lemma 2.6 (see [1]). *Let (X,d) be a cone metric space and P a normal cone with normal constant K. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.*

The following Corollary of Rezapour [23] will be needed in the sequel.

Corollary 2.7 (see [23]). *Let $a,b,c,u \in E$, the real Banach space.*

(i) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*

(ii) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

(iii) *If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.*

The following remarks of Radenović and Rhoades [10] will be needed in the sequel.

Remark 2.8 (see [10]). If $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.

Remark 2.9 (see [10]). If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$, where $\{x_n\}$ is a sequence and x is a given point in X.

Remark 2.10 (see [10]). If $0 \leq a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, then $a \leq b$ for each cone P.

Remark 2.11 (see [10]). If E is a real Banach space with a cone P and if $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

3. Main Results

In the following, we suppose that E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$, and $\leq$ is partial ordering with respect to P.

Theorem 3.1. *Let (X, d) be a complete cone metric space and M a nonempty closed subset of X such that for each $x \in M$ and $y \notin M$ there exists a point $z \in \partial M$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{3.1}$$

Suppose that $f, T : M \to X$ are two nonself mappings satisfying for all $x, y \in M$ with $x \neq y$,

$$\begin{aligned} &d(Tx,Ty) \leq \alpha d(fx,fy) + \beta u + \gamma v, \\ &\quad \text{where } u \in \{d(fx,Tx), d(fy,Ty)\}, \\ &\qquad v \in \{d(fx,Tx) + d(fy,Ty), d(fx,Ty) + d(fy,Tx)\}, \end{aligned} \tag{3.2}$$

and α, β, γ are nonnegative real numbers such that

$$\alpha + 2\beta + 3\gamma + \alpha\gamma < 1. \tag{3.3}$$

Also assume that

(i) $\partial M \subseteq fM$, $TM \cap M \subset fM$;

(ii) $fx \in \partial M \Rightarrow Tx \in M$;

(iii) *fM is closed in X;*

Then there exists a coincidence point of f and T in M. Moreover, if T and f are coincidentally commuting, then T and f have a unique common fixed point in M.

Proof. Two sequences $\{x_n\}$ and $\{y_n\}$ are constructed in the following way. Let $x \in \partial M$. As $\partial M \subseteq fM$, by (i) there exists a point $x_0 \in M$ such that $x = fx_0 \in \partial M$. Since $fx \in \partial M \Rightarrow Tx \subset M$, from (ii) it follows that $fx_0 \in \partial M \Rightarrow Tx_0 \in M \Rightarrow Tx_0 \in M \cap TM \subset fM$. Let $x_1 \in M$ be such that $y_1 = fx_1 = Tx_0 \in M$. Since $y_1 = Tx_0$, there exists $y_2 = Tx_1$ such that $d(y_1, y_2) = d(Tx_0, Tx_1)$.

If $y_2 \in M$, then $y_2 \in M \cap TM \subseteq fM$ which implies that there exists a point $x_2 \in M$ such that $y_2 = fx_2$. Otherwise, if $y_2 \notin M$, then there exists a point $u \in \partial M$ such that

$$d(fx_1, u) + d(u, y_2) = d(fx_1, y_2). \tag{3.4}$$

Since $u \in \partial M \subseteq fM$, there exists a point $x_2 \in M$ such that $u = fx_2$ and thus

$$d(fx_1, fx_2) + d(fx_2, y_2) = d(fx_1, y_2). \tag{3.5}$$

Assume that $y_3 = Tx_2$.

Thus repeating the arguments, two sequences $\{x_n\} \subseteq M$ and $\{y_n\} \subseteq TM \subset X$ are obtained such that

(i) $y_{n+1} = Tx_n$;

(ii) $y_n \in M \Rightarrow y_n = fx_n$;

(iii) $y_n \neq fx_n$ whenever $y_n \notin M$, then there exists $fx_n \in \partial M$ such that

$$d(fx_{n-1}, fx_n) + d(fx_n, y_n) = d(fx_{n-1}, y_n). \tag{3.6}$$

Next, we claim that $\{fx_n\}$ is a Cauchy sequence in fM. The following are derived. Let us denote

$$P = \{fx_i \in \{fx_n\} : fx_i = y_i\}, \qquad Q = \{fx_i \in \{fx_n\} : fx_i \neq y_i\}. \tag{3.7}$$

Obviously, two consecutive terms cannot lie in Q. Note that, if $fx_n \in Q$, then fx_{n-1} and fx_{n+1} belong to P. Now, three cases are distinguished.

Case 1. If $fx_n, fx_{n+1} \in P$, then $y_n = fx_n$, $y_{n+1} = fx_{n+1}$. Now from (3.2),

$$d(fx_n, fx_{n+1}) = d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha d(fx_{n-1}, fx_n) + \beta u + \gamma v, \tag{3.8}$$

where

$$\begin{aligned} u &\in \{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\ &= \{d(fx_{n-1}, y_n), d(fx_n, y_{n+1})\} \\ &= \{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}, \\ v &\in \{d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n), d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})\} \\ &= \{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1})\}. \end{aligned} \tag{3.9}$$

Thus

$$u \in \{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\},$$
$$v \in \{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1})\}. \tag{3.10}$$

Now four subcases arise.

Subcase 1.1. If $u = d(fx_{n-1}, fx_n)$ and $v = d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})$, then (3.8) becomes

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, fx_n) + \gamma[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})], \\ d(fx_n, fx_{n+1}) &\le \frac{\alpha+\beta+\gamma}{1-\gamma} d(fx_{n-1}, fx_n). \end{aligned} \tag{3.11}$$

Subcase 1.2. If $u = d(fx_{n-1}, fx_n)$ and $v = d(fx_{n-1}, fx_{n+1})$, then (3.8) becomes

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, fx_n) + \gamma[d(fx_{n-1}, fx_{n+1})] \\ &\le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, fx_n) + \gamma[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})], \\ d(fx_n, fx_{n+1}) &\le \frac{\alpha+\beta+\gamma}{1-\gamma} d(fx_{n-1}, fx_n). \end{aligned} \tag{3.12}$$

Subcase 1.3. If $u = d(fx_n, fx_{n+1})$ and $v = d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})$, then (3.8) becomes

$$\begin{gathered} d(fx_n, fx_{n+1}) \le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1}) + \gamma[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})], \\ d(fx_n, fx_{n+1}) \le \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{n-1}, fx_n). \end{gathered} \tag{3.13}$$

Subcase 1.4. If $u = d(fx_n, fx_{n+1})$ and $v = d(fx_{n-1}, fx_{n+1})$, then (3.8) becomes

$$\begin{aligned} d(fx_n, fx_{n+1}) &\le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1}) + \gamma[d(fx_{n-1}, fx_{n+1})], \\ &\le \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1}) + \gamma[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})], \\ d(fx_n, fx_{n+1}) &\le \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{n-1}, fx_n). \end{aligned} \tag{3.14}$$

Combining all Subcases 1.1, 1.2, 1.3, and 1.4, it follows that

$$d(fx_n, fx_{n+1}) \leq hd(fx_{n-1}, fx_n), \tag{3.15}$$

where $h = \max\{(\alpha+\beta+\gamma)/(1-\gamma), (\alpha+\gamma)/(1-\beta-\gamma)\} = (\alpha+\beta+\gamma)/(1-\gamma)$. Hence

$$d(fx_n, fx_{n+1}) = d(y_n, y_{n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\gamma} d(fx_{n-1}, fx_n). \tag{3.16}$$

Case 2. If $fx_n \in P$, $fx_{n+1} \in Q$, then $y_n = fx_n$, $y_{n+1} \neq fx_{n+1}$. Now,

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, y_{n+1}) \\ &= d(fx_n, y_{n+1}) \text{ by } (3.6) \\ &= d(y_n, y_{n+1}) \\ &= d(Tx_{n-1}, Tx_n). \end{aligned} \tag{3.17}$$

Proceeding as in Case 1,

$$d(fx_n, fx_{n+1}) \leq d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \frac{\alpha+\beta+\gamma}{1-\gamma} d(fx_{n-1}, fx_n). \tag{3.18}$$

Case 3. If $fx_n \in Q$, $fx_{n+1} \in P$, then $fx_{n-1} \in P$, $y_n \neq fx_n$, $y_{n+1} = fx_{n+1}, y_{n-1} = fx_{n-1}$ and $y_n = Tx_{n-1}$. Now,

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(fx_n, y_{n+1}) \\ &\leq d(fx_n, y_n) + d(y_n, y_{n+1}) \\ &= d(fx_n, y_n) + d(Tx_{n-1}, Tx_n) \\ &= d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta u + \gamma v. \end{aligned} \tag{3.19}$$

Thus

$$d(fx_n, fx_{n+1}) \leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta u + \gamma v, \tag{3.20}$$

where

$$\begin{aligned}
u &\in \{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n)\} \\
&= \{d(fx_{n-1}, y_n), d(fx_n, y_{n+1})\} \\
&= \{d(fx_{n-1}, y_n), d(fx_n, fx_{n+1})\}, \\
v &= \{d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n), d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})\} \\
&= \{d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)\}.
\end{aligned} \tag{3.21}$$

Thus

$$\begin{gathered}
u \in \{d(fx_{n-1}, y_n), d(fx_n, fx_{n+1})\}, \\
v \in \{d(fx_{n-1}, y_n) + d(fx_n, y_{n+1}), d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)\}.
\end{gathered} \tag{3.22}$$

Again four subcases arise.

Subcase 3.1. If $u = d(fx_{n-1}, y_n)$, $v = d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})$, then (3.20) becomes

$$\begin{aligned}
d(fx_n, fx_{n+1}) &= d(fx_n, y_{n+1}) \\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, y_n) \\
&\quad + \gamma[d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})] \\
&\leq [d(fx_{n-1}, y_n) - d(fx_{n-1}, fx_n)] + \alpha d(fx_{n-1}, fx_n) + (\beta + \gamma) d(fx_{n-1}, y_n) \\
&\quad + \gamma d(fx_n, fx_{n+1}) \text{ by } (3.6) \\
&\leq \frac{1 + \beta + \gamma}{1 - \gamma} d(fx_{n-1}, y_n) + \frac{\alpha - 1}{1 - \gamma} d(fx_{n-1}, fx_n) \\
&= \frac{1 + \beta + \gamma + \alpha - 1}{1 - \gamma} d(fx_{n-1}, y_n).
\end{aligned} \tag{3.23}$$

Thus

$$d(fx_n, fx_{n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \gamma} d(fx_{n-1}, y_n), \tag{3.24}$$

where using Case 2,

$$d(fx_{n-1}, y_n) = d(y_{n-1}, y_n) = d(Tx_{n-2}, Tx_{n-1}) \leq \frac{\alpha + \beta + \gamma}{1 - \gamma} d(fx_{n-2}, fx_{n-1}). \tag{3.25}$$

Then (3.24) becomes

$$d(fx_n, fx_{n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right]^2 d(fx_{n-2}, fx_{n-1}). \tag{3.26}$$

Subcase 3.2. If $u = d(fx_{n-1}, y_n)$, $v = d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)$, then (3.20) becomes

$$\begin{aligned}
d(fx_n, fx_{n+1}) &= d(fx_n, y_{n+1}) \\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, y_n) \\
&\quad + \gamma\left[d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)\right] \\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, y_n) \\
&\quad + \gamma\left[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) + d(fx_n, y_n)\right] \\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_{n-1}, y_n) \\
&\quad + \gamma\left[d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})\right] \text{ by } (3.6).
\end{aligned} \tag{3.27}$$

Proceeding as in Subcase 3.1, it follows that

$$d(fx_n, fx_{n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right]^2 d(fx_{n-2}, fx_{n-1}). \tag{3.28}$$

Subcase 3.3. If $u = d(fx_n, fx_{n+1})$, $v = d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})$, then (3.20) becomes

$$\begin{aligned}
d(fx_n, fx_{n+1}) &= d(fx_n, y_{n+1}) \\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1}) \\
&\quad + \gamma\left[d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})\right] \\
&\leq \left[d(fx_{n-1}, y_n) - d(fx_{n-1}, fx_n)\right] + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1}) \\
&\quad + \gamma\left[d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})\right] \\
&\leq \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{n-1}, y_n).
\end{aligned} \tag{3.29}$$

Thus

$$d(fx_n, fx_{n+1}) \leq \frac{\alpha+\gamma}{1-\beta-\gamma} d(fx_{n-1}, y_n), \tag{3.30}$$

where using Case 2,

$$d(fx_{n-1}, y_n) = d(y_{n-1}, y_n) = d(Tx_{n-2}, Tx_{n-1}) \leq \frac{\alpha+\beta+\gamma}{1-\gamma} d(fx_{n-2}, fx_{n-1}). \tag{3.31}$$

Then (3.30) becomes

$$d(fx_n, fx_{n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right], \left[\frac{\alpha+\gamma}{1-\beta-\gamma}\right] d(fx_{n-2}, fx_{n-1}). \tag{3.32}$$

Subcase 3.4. If $u = d(fx_n, fx_{n+1})$, $v = d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)$, then (3.20) becomes

$$\begin{aligned}
d(fx_n, fx_{n+1}) = d(fx_n, y_{n+1}) &\\
&\leq d(fx_n, y_n) + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1})\\
&\quad + \gamma\left[d(fx_{n-1}, y_{n+1}) + d(fx_n, y_n)\right]\\
&\leq \left[d(fx_{n-1}, y_n) - d(fx_{n-1}, fx_n)\right] + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1})\\
&\quad + \gamma\left[d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) + d(fx_n, y_n)\right]\\
&= \left[d(fx_{n-1}, y_n) - d(fx_{n-1}, fx_n)\right] + \alpha d(fx_{n-1}, fx_n) + \beta d(fx_n, fx_{n+1})\\
&\quad + \gamma\left[d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1})\right].
\end{aligned} \tag{3.33}$$

Proceeding as in Subcase 3.1, it follows that

$$d(fx_n, fx_{n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right], \left[\frac{\alpha+\gamma}{1-\beta-\gamma}\right] d(fx_{n-2}, fx_{n-1}). \tag{3.34}$$

Combining all four Subcases 3.1, 3.2, 3.3, and 3.4, we have

$$d(fx_n, fx_{n+1}) \leq k d(fx_{n-2}, fx_{n-1}), \tag{3.35}$$

where $k = \max\{[(\alpha+\beta+\gamma)/(1-\gamma)]^2, [(\alpha+\beta+\gamma)/(1-\gamma)], [(\alpha+\gamma)/(1-\beta-\gamma)]\} = [(\alpha+\beta+\gamma)/(1-\gamma)]^2 < 1$ by (3.3). Hence

$$d(fx_n, fx_{n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right]^2 d(fx_{n-2}, fx_{n-1}). \tag{3.36}$$

Now, combining main Cases 1, 2, and 3, it follows that

$$d(fx_n, fx_{n+1}) \leq \mu w_n, \tag{3.37}$$

where

$$\begin{aligned}\mu &= \max\left\{\frac{\alpha+\beta+\gamma}{1-\gamma}, \left[\frac{\alpha+\beta+\gamma}{1-\gamma}\right]^2\right\}\\ &= \frac{\alpha+\beta+\gamma}{1-\gamma} \quad \text{as } \frac{\alpha+\beta+\gamma}{1-\gamma} < 1 \text{ by (3.3)},\\ w_n &\in \{d(fx_{n-1}, fx_n), d(fx_{n-2}, fx_{n-1})\}.\end{aligned} \tag{3.38}$$

Following the procedure of Assad and Kirk [12], it can be easily shown by induction that for $n > 1$,

$$d(fx_n, fx_{n+1}) \leq \mu^{(n-1)/2} w_2, \quad \text{where } w_2 = \{d(fx_2, fx_1), d(fx_0, fx_1)\}. \tag{3.39}$$

By triangle inequality, for $n > m$, it follows that

$$\begin{aligned}d(fx_n, fx_m) &\leq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m)\\ &\leq \left(\mu^{(n-1)/2} + \mu^{(n-2)/2} + \cdots + \mu^{(m-1)/2}\right) w_2\\ &\leq \frac{\sqrt{\mu^{n-1}}}{1-\sqrt{\mu}} w_2 \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.\end{aligned} \tag{3.40}$$

From Remark 2.9, $d(fx_n, fx_m) \ll c$, which implies by Definition 2.4(2) that $\{fx_n\}$ is a Cauchy sequence in fM which is a closed subset of the complete cone metric space and hence is complete. Then there exists a point $z \in M \cap fM$ such that $fx_n \to z$ as $n \to \infty$. Thus

$$d(fx_n, z) \ll c \quad \text{for sufficiently large } n. \tag{3.41}$$

Since $z \in fM$, there exists a point $w \in M$ such that $z = fw$. By the construction of $\{fx_n\}$, it was seen that there exists a subsequence $\{fx_m\}$ such that

$$y_m = fx_m = Tx_{m-1}. \tag{3.42}$$

We will prove that $Tw = z$. Consider

$$\begin{aligned}d(Tw, z) &\leq d(Tw, Tx_{m-1}) + d(Tx_{m-1}, z)\\ &\leq [\alpha d(fw, fx_{m-1}) + \beta u + \gamma v] + d(fx_m, z)\\ &= [\alpha d(z, fx_{m-1}) + \beta u + \gamma v] + d(fx_m, z),\end{aligned} \tag{3.43}$$

where

$$\begin{aligned} u &\in \{d(fw,Tw), d(fx_{m-1},Tx_{m-1})\} \\ &= \{d(z,Tw), d(fx_{m-1},fx_m)\}, \\ v &\in \{d(fw,Tw)+d(fx_{m-1},Tx_{m-1}), d(fw,Tx_{m-1})+d(fx_{m-1},Tw)\} \\ &= \{d(z,Tw)+d(fx_{m-1},fx_m), d(z,fx_m)+d(fx_{m-1},Tw)\}. \end{aligned} \tag{3.44}$$

Thus

$$\begin{gathered} u \in \{d(z,Tw), d(fx_{m-1},fx_m)\}, \\ v \in \{d(z,Tw)+d(fx_{m-1},fx_m), d(z,fx_m)+d(fx_{m-1},Tw)\}. \end{gathered} \tag{3.45}$$

Now again four cases arise.

Case 1. If $u = d(z,Tw)$, $v = d(z,Tw)+d(fx_{m-1},fx_m)$, then (3.43) becomes

$$\begin{aligned} d(Tw,z) &\le \alpha d(z,fx_{m-1}) + \beta d(z,Tw) + \gamma[d(z,Tw)+d(fx_{m-1},fx_m)] + d(fx_m,z) \\ &\le \alpha d(z,fx_{m-1}) + \beta d(z,Tw) + \gamma[d(z,Tw)+d(fx_{m-1},z)+d(z,fx_m)] + d(fx_m,z) \\ &= \frac{\alpha+\gamma}{1-\beta-\gamma} d(z,fx_{m-1}) + \frac{1+\gamma}{1-\beta-\gamma} d(fx_m,z). \end{aligned} \tag{3.46}$$

Case 2. If $u = d(z,Tw)$, $v = d(z,fx_m)+d(fx_{m-1},Tw)$, then (3.43) becomes

$$\begin{aligned} d(Tw,z) &\le \alpha d(z,fx_{m-1}) + \beta d(z,Tw) + \gamma[d(z,fx_m)+d(fx_{m-1},Tw)] + d(fx_m,z) \\ &\le \alpha d(z,fx_{m-1}) + \beta d(z,Tw) + \gamma[d(z,fx_m)+d(fx_{m-1},z)+d(z,Tw)] + d(fx_m,z) \\ &= \frac{\alpha+\gamma}{1-\beta-\gamma} d(z,fx_{m-1}) + \frac{1+\gamma}{1-\beta-\gamma} d(fx_m,z). \end{aligned} \tag{3.47}$$

Case 3. If $u = d(fx_{m-1},z)+d(z,fx_m)$, $v = d(z,Tw)+d(fx_{m-1},fx_m)$, then (3.43) becomes

$$\begin{aligned} d(Tw,z) &\le \alpha d(z,fx_{m-1}) + \beta[d(fx_{m-1},z)+d(z,fx_m)] \\ &\quad + \gamma[d(z,Tw)+d(fx_{m-1},fx_m)] + d(fx_m,z) \\ &\le \alpha d(z,fx_{m-1}) + \beta[d(fx_{m-1},z)+d(z,fx_m)] \\ &\quad + \gamma[d(z,Tw)+d(fx_{m-1},z)+d(z,fx_m)] + d(fx_m,z) \\ &= \frac{\alpha+\beta+\gamma}{1-\gamma} d(z,fx_{m-1}) + \frac{1+\beta+\gamma}{1-\gamma} d(fx_m,z). \end{aligned} \tag{3.48}$$

Case 4. If $u = d(fx_{m-1}, z) + d(z, fx_m)$, $v = d(z, fx_m) + d(fx_{m-1}, Tw)$, then (3.43) becomes

$$\begin{aligned} d(Tw, z) &\leq \alpha d(z, fx_{m-1}) + \beta\left[d(fx_{m-1}, z) + d(z, fx_m)\right] \\ &\quad + \gamma\left[d(z, fx_m) + d(fx_{m-1}, Tw)\right] + d(fx_m, z) \\ &\leq \alpha d(z, fx_{m-1}) + \beta\left[d(fx_{m-1}, z) + d(z, fx_m)\right] \\ &\quad + \gamma\left[d(z, Tw) + d(fx_{m-1}, z) + d(z, fx_m)\right] + d(fx_m, z) \\ &= \frac{\alpha+\beta+\gamma}{1-\gamma} d(z, fx_{m-1}) + \frac{1+\beta+\gamma}{1-\gamma} d(fx_m, z). \end{aligned} \tag{3.49}$$

Combining Cases 1, 2, 3, and 4, it follows that

$$\begin{aligned} d(Tw, z) &\leq \max\left\{\frac{\alpha+\gamma}{1-\beta-\gamma}, \frac{\alpha+\beta+\gamma}{1-\gamma}\right\} d(z, fx_{m-1}) \\ &\quad + \max\left\{\frac{1+\gamma}{1-\beta-\gamma}, \frac{1+\beta+\gamma}{1-\gamma}\right\} d(fx_m, z) \\ &= \frac{\alpha+\beta+\gamma}{1-\gamma} d(z, fx_{m-1}) + \frac{1+\beta+\gamma}{1-\gamma} d(fx_m, z). \end{aligned} \tag{3.50}$$

Thus

$$d(Tw, z) \leq k d(z, fx_{m-1}) + k' d(fx_m, z), \tag{3.51}$$

where $k = (\alpha+\beta+\gamma)/(1-\gamma)$, $k' = (1+\beta+\gamma)/(1-\gamma)$.

Let $c \in E$ be given with $0 \ll c$. From $fx_{m-1} \to z$ as $m \to \infty$ and Definition 2.4(1),

$$d(fx_{m-1}, z) \ll \frac{c}{3k} \quad \forall n > m \geq N_2. \tag{3.52}$$

From $Tx_{m-1} \to z$ as $m \to \infty$ and by Definition 2.4 (1),

$$d(fx_m, z) = d(Tx_{m-1}, z) \ll \frac{c}{3k'} \quad \forall n > m \geq N_3. \tag{3.53}$$

From the definition of convergence in cone metric space and by (3.52) and (3.53), inequality (3.43) becomes

$$d(Tw, z) \ll \frac{kc}{2k} + \frac{k'c}{2k'} = \frac{c}{2} + \frac{c}{2} = c. \tag{3.54}$$

Therefore, $d(Tw, z) \ll c$ for each $c \in \operatorname{int} P$. Then by (iii) of Corollary 2.7, we have $d(Tw, z) = \mathbf{0}$, that is, $Tw = z = fw$ which implies that w is the coincidence point of f and T.

Since T and f are coincidentally commuting, $Tfw = fTw$ for $w \in C(f,T)$ which implies $Tz = fz$. Consider

$$\begin{aligned} d(Tz,z) &= d(Tz,Tw) \\ &\leq \alpha d(fz,fw) + \beta u + \gamma v \\ &\leq \alpha d(Tz,z) + \beta u + \gamma v, \end{aligned} \tag{3.55}$$

where

$$\begin{aligned} u &\in \{d(fz,Tz), d(fw,Tw)\} = \{\mathbf{0},\mathbf{0}\} = \{\mathbf{0}\}. \\ v &\in \{d(fz,Tz) + d(fw,Tw), d(fz,Tw) + d(fw,Tz)\} \\ &= \{\mathbf{0}, d(Tz,z) + d(z,Tz)\} = \{\mathbf{0}, 2d(Tz,z)\}. \end{aligned} \tag{3.56}$$

Thus $u \in \{\mathbf{0}\}$ and $v \in \{\mathbf{0}, 2d(Tz,z)\}$. Two cases arise.

Case 1. If $u = \mathbf{0}$ and $v = \mathbf{0}$, then (3.55) becomes

$$\begin{aligned} d(Tz,z) &\leq \alpha d(Tz,z) + \beta\mathbf{0} + \gamma\mathbf{0} \\ &\leq \alpha d(Tz,z). \end{aligned} \tag{3.57}$$

Case 2. If $u = 0$ and $v = 2d(Tz,z)$, then (3.55) becomes

$$\begin{aligned} d(Tz,z) &\leq \alpha d(Tz,z) + \beta 0 + \gamma d(Tz,z) \\ &\leq (\alpha + 2\gamma) d(Tz,z). \end{aligned} \tag{3.58}$$

Combining Cases 1 and 2, it follows that

$$d(Tz,z) \leq \max\{\alpha + 2\gamma, \alpha\} d(Tz,z) = (\alpha + 2\gamma) d(Tz,z). \tag{3.59}$$

Since $\alpha + 2\gamma \leq \alpha + 2\beta + 3\gamma + \alpha\gamma < 1$ by (3.3), it follows from Remark 2.11 that $d(Tz,z) = \mathbf{0}$ which implies that $Tz = z$. Thus $fz = Tz = z$.

Uniqueness: if $p \neq z$ is another common fixed point of f and T in M, then $p = Tp = fp$. Now by (3.2), it follows that

$$\begin{aligned} d(T\mathrm{p},Tz) &\leq \alpha d(fp,fz) + \beta u + \gamma v, \\ d(p,z) &\leq \alpha d(p,z) + \beta u + \gamma v, \end{aligned} \tag{3.60}$$

where

$$u \in \{d(fp,Tp), d(fz,Tz)\} = \{0\},$$
$$v \in \{d(fp,Tp) + d(fz,Tz), d(fp,Tz) + d(fz,Tp)\} = \{0, 2d(p,z)\}. \tag{3.61}$$

Thus $u = \mathbf{0}$ and $v \in \{\mathbf{0}, 2d(p,z)\}$. Two cases arise.

Case 1. If $u = \mathbf{0}$ and $v = \mathbf{0}$, then (3.60) becomes $d(p,z) \leq \alpha d(p,z)$. Since $\alpha < \alpha + 2\beta + 3\gamma + \alpha\gamma < 1$, by Remark 2.11 we have $d(p,z) = 0$ which implies that $z = p$ is the unique common fixed point of f and T.

Case 2. If $u = \mathbf{0}$ and $v = 2d(p,z)$, then (3.60) becomes

$$d(p,z) \leq \alpha d(p,z) + \gamma(2d(p,z)) = (\alpha + 2\gamma)d(p,z) \tag{3.62}$$

Since $\alpha + 2\gamma < \alpha + 2\beta + 3\gamma + \alpha\gamma < 1$, by Remark 2.11 we have $d(p,z) = 0$ which implies $z = p$ is the unique common fixed point of f and T. Hence $fz = Tz = z$ is the unique common fixed point of f and T in M. □

The following example illustrates Theorem 3.1.

Example 3.2. Let $E = \Re^2$, $P = \{(x,y) : x \geq 0,\ y \geq 0\}$, $X = [0,\infty)$, $d(x,y) = (|x-y|, k|x-y|)$, $k \geq 0$ and $M = [0,1/3]$. Define two nonself mappings $T, f : M \to X$ as $Tx = 2x/(1+2x)$ and $fx = 2x$ for all $x \in M$.

Now let us see that conditions (i)–(iii) in Theorem 3.1 are satisfied.

It may be seen that $\partial M = \{0, 1/3\}$, $TM = [0,2/5]$, and $fM = [0,2/3]$. Then $\partial M \subset fM$ and $TM \cap M = [0,2/5] \cap [0,1/3] = [0,1/3] \subset fM$. Also, $fx \in \partial M \Rightarrow Tx \in M$ as $f0 = 0 \in \partial M \Rightarrow T0 = 0 \in M$. Moreover fM is closed in X.

Next, we shall see that inequality (3.2) is satisfied by taking $\alpha = 2/3$ and $\beta = \gamma = 1/24$. It is easy to see that $\alpha + 2\beta + 3\gamma + \alpha\gamma < 1$.

Now, LHS of inequality (3.2) is $d(Tx,Ty) = (|Tx-Ty|, k|Tx-Ty|)$. Taking $x = 1/3$ and $y = 1/6$, it follows that $d(Tx,Ty) = (0.15, 0.15k)$.

Next, RHS of inequality (3.2) is $\alpha d(fx,fy) + \beta u + \gamma v$, where $d(fx,fy) = (0.334, 0.334k)$, $u = \{(0.26, 0.26k), (0.084, 0.084k)\}$, and $v = \{(0.34, 0.34k), (0.476, 0.476k)\}$. Then RHS of inequality (3.2) is $(0.248, 0.248k)$ if $u = (0.26, 0.26k)$ and $v = (0.34, 0.34k)$. Thus LHS of inequality (3.2) < RHS of inequality (3.2). Similarly, LHS of inequality (3.2) < RHS of inequality (3.2) for all possible cases of u and v. Thus all the conditions of Theorem 3.1 are satisfied. Hence "0" is the unique common fixed point of f and T in M.

Corollary 3.3. *Let (X,d) be a complete cone metric space and M a nonempty closed subset of X such that for each $x \in M$ and $y \notin M$ there exists a point $z \in \partial M$ such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{3.63}$$

Suppose that $T : M \to X$ is a nonself mapping satisfying for all $x, y \in M$ with $x \neq y$,

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta u + \gamma v \\ \text{where } u &\in \{d(x, Tx), d(y, Ty)\}, \\ v &\in \{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\}, \end{aligned} \tag{3.64}$$

and α, β, γ are nonnegative real numbers such that $\alpha + 2\beta + 3\gamma + \alpha\gamma < 1$. Also assume that $x \in \partial M \Rightarrow Tx \in M$. Then there exists a unique fixed point of T in M.

Proof. The proof of this corollary follows by taking $f = I_X$, the identity mapping of X in Theorem 3.1. □

Remark 3.4. Our results generalize the results of Radenović and Rhoades [10] and Janković et al. [6] and extend the results of Ćirić and Ume [20] to cone metric space for single valued mappings.

Acknowledgment

The authors would like to thank the referees for their valuable suggestions which lead to the improvement of the presentation of the paper.

References

[1] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.
[2] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.
[3] M. Abbas and B. E. Rhoades, "Fixed and periodic point results in cone metric spaces," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 511–515, 2009.
[4] D. Ilić and V. Rakočević, "Common fixed points for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 876–882, 2008.
[5] P. Raja and S. M. Vaezpour, "Some extensions of Banach's contraction principle in complete cone metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 768294, 11 pages, 2008.
[6] S. Janković, Z. Kadelburg, S. Radenović, and B. E. Rhoades, "Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 761086, 16 pages, 2009.
[7] G. Jungck, S. Radenović, S. Radojević, and V. Rakočević, "Common fixed point theorems for weakly compatible pairs on cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 643840, 13 pages, 2009.
[8] Z. Kadelburg, S. Radenović, and V. Rakočević, "Remarks on "Quasi-contraction on a cone metric space"," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1674–1679, 2009.
[9] Z. Kadelburg, S. Radenović, and B. Rosić, "Strict contractive conditions and common fixed point theorems in cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 173838, 14 pages, 2009.
[10] S. Radenović and B. E. Rhoades, "Fixed point theorem for two non-self mappings in cone metric spaces," *Computers & Mathematics with Applications*, vol. 57, no. 10, pp. 1701–1707, 2009.
[11] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.

[12] N. A. Assad and W. A. Kirk, "Fixed point theorems for set-valued mappings of contractive type," *Pacific Journal of Mathematics*, vol. 43, no. 3, pp. 553–562, 1972.
[13] N. A. Assad, "On a fixed point theorem of Kannan in Banach spaces," *Tamkang Journal of Mathematics*, vol. 7, no. 1, pp. 91–94, 1976.
[14] L. Ćirić, "Non-self mappings satisfying non-linear contractive condition with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2927–2935, 2009.
[15] O. Hadžić, "On coincidence points in convex metric spaces," *Univerzitet u Novom Sadu. Zbornik Radova Prirodno-Matematičkog Fakulteta. Serija za Matematiku*, vol. 19, no. 2, pp. 233–240, 1989.
[16] O. Hadžić and L. Gajić, "Coincidence points for set-valued mappings in convex metric spaces," *Univerzitet u Novom Sadu. Zbornik Radova Prirodno-Matematičkog Fakulteta. Serija za Matematiku*, vol. 16, no. 1, pp. 13–25, 1986.
[17] M. Imdad and S. Kumar, "Rhoades-type fixed-point theorems for a pair of nonself mappings," *Computers & Mathematics with Applications*, vol. 46, no. 5-6, pp. 919–927, 2003.
[18] B. E. Rhoades, "A fixed point theorem for some non-self-mappings," *Mathematica Japonica*, vol. 23, no. 4, pp. 457–459, 1978.
[19] B. E. Rhoades, "A fixed point theorem for non-self set-valued mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 20, no. 1, pp. 9–12, 1997.
[20] L. B. Ćirić and J. S. Ume, "Multi-valued non-self-mappings on convex metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 6, pp. 1053–1063, 2005.
[21] S. Itoh, "Multivalued generalized contractions and fixed point theorems," *Commentationes Mathematicae Universitatis Carolinae*, vol. 18, no. 2, pp. 247–258, 1977.
[22] M. S. Khan, "Common fixed point theorems for multivalued mappings," *Pacific Journal of Mathematics*, vol. 95, no. 2, pp. 337–347, 1981.
[23] Sh. Rezapour, "A review on topological properties of cone metric spaces," in *Proceedings of the Conference on Analysis, Topology and Applications (ATA '08)*, Vrnjacka Banja, Serbia, May-June 2008.