Research Article

Strong Convergence Theorems of a New General Iterative Process with Meir-Keeler Contractions for a Countable Family of $\lambda_i$-Strict Pseudocontractions in $q$-Uniformly Smooth Banach Spaces

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We introduce a new iterative scheme with Meir-Keeler contractions for strict pseudocontractions in $q$-uniformly smooth Banach spaces. We also discuss the strong convergence theorems for the new iterative scheme in $q$-uniformly smooth Banach space. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Throughout this paper, we denote by $E$ and $E^*$ a real Banach space and the dual space of $E$, respectively. Let $C$ be a subset of $E$, and $I_r T$ be a non-self-mapping of $C$. We use $F(T)$ to denote the set of fixed points of $T$.

The norm of a Banach space $E$ is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y$ on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. If, for each $y \in S(E)$, the limit (1.1) is uniformly attained for $x \in S(E)$, then the norm of $E$ is said to be uniformly Gâteaux differentiable. The norm of $E$ is said to be Fréchet differentiable if, for each $x \in S(E)$, the limit (1.1) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be uniformly Fréchet differentiable (or uniformly smooth) if the limit (1.1) is attained uniformly for $x, y \in S(E) \times S(E)$. 
Let \( \rho_E : [0, 1) \to [0, 1) \) be the modulus of smoothness of \( E \) defined by

\[
\rho_E(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(E) \text{, } \|y\| \leq t \right\}. \tag{1.2}
\]

A Banach space \( E \) is said to be uniformly smooth if \( \rho_E(t)/t \to 0 \) as \( t \to 0 \). Let \( q > 1 \). A Banach space \( E \) is said to be \( q \)-uniformly smooth, if there exists a fixed constant \( c > 0 \) such that \( \rho_E(t) \leq ct^q \). It is well known that \( E \) is uniformly smooth if and only if the norm of \( E \) is uniformly Fréchet differentiable. If \( E \) is \( q \)-uniformly smooth, then \( q \leq 2 \) and \( E \) is uniformly smooth, and hence the norm of \( E \) is uniformly Fréchet differentiable, in particular, the norm of \( E \) is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are \( L^p \), where \( p > 1 \). More precisely, \( L^p \) is \( \min \{p, 2\} \)-uniformly smooth for every \( p > 1 \).

By a gauge we mean a continuous strictly increasing function \( \varphi \) defined \( \mathbb{R}^+ : = [0, \infty) \) such that \( \varphi(0) = 0 \) and \( \lim_{r \to \infty} \varphi(r) = \infty \). We associate with a gauge \( \varphi \) a (generally multivalued) duality map \( J_\varphi : E \to E^* \) defined by

\[
J_\varphi(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|) \right\}. \tag{1.3}
\]

In particular, the duality mapping with gauge function \( \varphi(t) = t^{q-1} \) denoted by \( J_q \), is referred to the (generalized) duality mapping. The duality mapping with gauge function \( \varphi(t) = t \) denoted by \( J \), is referred to the normalized duality mapping. Browder [1] initiated the study \( J_\varphi \). Set for \( t \geq 0 \)

\[
\Phi(t) = \int_0^t \varphi(r) \, dr. \tag{1.4}
\]

Then it is known that \( J_\varphi(x) \) is the subdifferential of the convex function \( \Phi(\| \cdot \|) \) at \( x \). It is well known that if \( E \) is smooth, then \( J_q \) is single valued, which is denoted by \( j_q \).

The duality mapping \( J_q \) is said to be weakly sequentially continuous if the duality mapping \( J_q \) is single valued and for any \( \{x_n\} \in E \) with \( x_n \rightharpoonup x \), \( J_q(x_n) \rightharpoonup J_q(x) \). Every \( L^p \) (\( 1 < p < \infty \)) space has a weakly sequentially continuous duality map with the gauge \( \varphi(t) = t^{p-1} \). Gossez and Lami Dozo [2] proved that a space with a weakly continuous duality mapping satisfies Opial’s condition. Conversely, if a space satisfies Opial’s condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous duality mapping. We already know that in \( q \)-uniformly smooth Banach space, there exists a constant \( C_q > 0 \) such that

\[
\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + C_q \|y\|^q, \tag{1.5}
\]

for all \( x, y \in E \).

Recall that a mapping \( T \) is said to be nonexpansive, if

\[
\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.6}
\]
$T$ is said to be a $\lambda$-strict pseudocontraction in the terminology of Browder and Petryshyn [3], if there exists a constant $\lambda > 0$ such that

$$
\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^2 - \lambda \| (I - T)x - (I - T)y \|^2,
$$

(1.7)

for every $x, y$, and $C$ for some $j_q(x - y) \in J_q(x - y)$. It is clear that (1.7) is equivalent to the following:

$$
\langle (I - T)x - (I - T)y, j_q(x - y) \rangle \geq \lambda \| (I - T)x - (I - T)y \|^2.
$$

(1.8)

The following famous theorem is referred to as the Banach contraction principle.

**Theorem 1.1 (Banach [4]).** Let $(X, d)$ be a complete metric space and let $f$ be a contraction on $X$, that is, there exists $r \in (0, 1)$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Then $f$ has a unique fixed point.

**Theorem 1.2 (Meir and Keeler [5]).** Let $(X, d)$ be a complete metric space and let $\phi$ be a Meir-Keeler contraction (MKC, for short) on $X$, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(\phi(x), \phi(y)) < \varepsilon$ for all $x, y \in X$. Then $\phi$ has a unique fixed point.

This theorem is one of generalizations of Theorem 1.1, because contractions are Meir-Keeler contractions.

In a smooth Banach space, we define an operator $A$ is strongly positive if there exists a constant $\gamma > 0$ with the property

$$
\langle Ax, J(x) \rangle \geq \gamma \|x\|^2, \quad \|aI - bA\| = \sup_{\|x\| \leq 1} \| (aI - bA)x, J(x) \| : a \in [0, 1], \ b \in [0, 1],
$$

(1.9)

where $I$ is the identity mapping and $J$ is the normalized duality mapping.

Attempts to modify the normal Mann’s iteration method for nonexpansive mappings and $\lambda$-strictly pseudocontractions so that strong convergence is guaranteed have recently been made; see, for example, [6–11] and the references therein.

Kim and Xu [6] introduced the following iteration process:

$$
x_1 = x \in C,
\begin{align*}
y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \\
x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0,
\end{align*}
$$

(1.10)

where $T$ is a nonexpansive mapping of $C$ into itself $u \in C$ is a given point. They proved the sequence $\{x_n\}$ defined by (1.10) converges strongly to a fixed point of $T$, provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.
Hu and Cai [12] introduced the following iteration process:

\[
x_1 = x \in C,
\]
\[
y_n = P_C \left[ \beta_n x_n + (1 - \beta_n) \sum_{i=1}^{N} \eta_i^{(n)} T_i x_n \right],
\]
\[
x_{n+1} = \alpha_n \gamma f(x_n) + \gamma_n x_n + \left[ (1 - \gamma_n) I - \alpha_n A \right] y_n, \quad n \geq 1.
\]

where \( T_i \) is non-self-\( \lambda_i \)-strictly pseudocontraction, \( f \) is a contraction and \( A \) is a strong positive linear bounded operator in Banach space. They have proved, under certain appropriate assumptions on the sequences \( \{\alpha_n\}, \{\gamma_n\}, \) and \( \{\beta_n\} \), that \( \{x_n\} \) defined by (1.11) converges strongly to a common fixed point of a finite family of \( \lambda_i \)-strictly pseudocontractions, which solves some variational inequality.

**Question 1.** Can Theorem 3.1 of Zhou [8], Theorem 2.2 of Hu and Cai [12] and so on be extended from finite \( \lambda_i \)-strictly pseudocontraction to infinite \( \lambda_i \)-strictly pseudocontraction?

**Question 2.** We know that the Meir-Keeler contraction (MKC, for short) is more general than the contraction. What happens if the contraction is replaced by the Meir-Keeler contraction?

The purpose of this paper is to give the affirmative answers to these questions mentioned above. In this paper we study a general iterative scheme as follows:

\[
x_1 = x \in C,
\]
\[
y_n = P_C \left[ \beta_n x_n + (1 - \beta_n) \sum_{i=1}^{\infty} \eta_i^{(n)} T_i x_n \right],
\]
\[
x_{n+1} = \alpha_n \gamma \phi(x_n) + \gamma_n x_n + \left[ (1 - \gamma_n) I - \alpha_n A \right] y_n, \quad n \geq 1,
\]

where \( T_n \) is non-self \( \lambda_n \)-strictly pseudocontraction, \( \phi \) is a MKC contraction and \( A \) is a strong positive linear bounded operator in Banach space. Under certain appropriate assumptions on the sequences \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \) and \( \{\mu_n\} \), that \( \{x_n\} \) defined by (1.12) converges strongly to a common fixed point of an infinite family of \( \lambda_i \)-strictly pseudocontractions, which solves some variational inequality.

### 2. Preliminaries

In order to prove our main results, we need the following lemmas.

**Lemma 2.1** (see [13]). Let \( \{x_n\}, \{z_n\} \) be bounded sequences in a Banach space \( E \) and \( \{\beta_n\} \) be a sequence in \([0, 1]\) which satisfies the following condition: \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose that \( x_{n+1} = (1 - \beta_n) x_n + \beta_n z_n \) for all \( n \geq 0 \) and \( \limsup_{n \to \infty} (\|z_{n+1} - z_n\| + \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|z_n - x_n\| = 0 \).
Lemma 2.2 (see Xu [14]). Assume that \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that 

\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,
\]

where \( \gamma_n \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence in \(\mathbb{R} \) such that

(i) \( \sum_{n=1}^{\infty} \gamma_n = \infty \),

(ii) \( \limsup_{n \to \infty} (\delta_n / \gamma_n) \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \to \infty} \alpha_n = 0 \).

Lemma 2.3 (see [15] demiclosedness principle). Let \( C \) be a nonempty closed convex subset of a reflexive Banach space \( E \) which satisfies Opial’s condition, and suppose \( T : C \to E \) is nonexpansive. Then the mapping \( I - T \) is demiclosed at zero, that is, \( x_n \to x, x_n - Tx_n \to 0 \) implies \( x = Tx \).

Lemma 2.4 (see [16, Lemmas 3.1, 3.3]). Let \( E \) be real smooth and strictly convex Banach space, and \( C \) be a nonempty closed convex subset of \( E \) which is also a sunny nonexpansive retraction of \( E \). Assume that \( T : C \to E \) is a nonexpansive mapping and \( P \) is a sunny nonexpansive retraction of \( E \) onto \( C \), then \( F(T) = F(PT) \).

Lemma 2.5 (see [17, Lemma 2.2]). Let \( C \) be a nonempty convex subset of a real \( q \)-uniformly smooth Banach space \( E \) and \( T : C \to C \) be a \( \lambda \)-strict pseudocontraction. For \( \alpha \in (0, 1) \), we define \( T_\alpha x = (1 - \alpha)x + \alpha Tx \). Then, as \( \alpha \in (0, \mu) \), \( \mu = \min\{1, [q\lambda/C_4]^{1/(q-1)}\} \), \( T_\alpha : C \to C \) is nonexpansive such that \( F(T_\alpha) = F(T) \).

Lemma 2.6 (see [12, Remark 2.6]). When \( T \) is non-self-mapping, the Lemma 2.5 also holds.

Lemma 2.7 (see [12, Lemma 2.8]). Assume that \( A \) is a strongly positive linear bounded operator on a smooth Banach space \( E \) with coefficient \( \gamma > 0 \) and \( 0 < \rho \leq \|A\|^{-1} \). Then,

\[
\|I - \rho A\| \leq 1 - \rho \gamma.
\]  

(2.1)

Lemma 2.8 (see [18, Lemma 2.3]). Let \( \phi \) be an MKC on a convex subset \( C \) of a Banach space \( E \). Then for each \( \epsilon > 0 \), there exists \( r \in (0, 1) \) such that

\[
\|x - y\| \geq \epsilon \text{ implies } \|\phi x - \phi y\| \leq r\|x - y\| \quad \forall x, y \in C.
\]  

(2.2)

Lemma 2.9. Let \( C \) be a closed convex subset of a reflexive Banach space \( E \) which admits a weakly sequentially continuous duality mapping \( J_q \) from \( E \) to \( E^* \). Let \( T : C \to C \) be a nonexpansive mapping with \( F(T) \neq \emptyset \) and \( \phi : C \to C \) be a MKC, \( A \) is strongly positive linear bounded operator with coefficient \( \gamma > 0 \). Assume that \( 0 < \gamma < \gamma \). Then the sequence \( \{x_t\} \) define by \( x_t = ty(x_t) + (1 - tA)Tx_t \) converges strongly as \( t \to 0 \) to a fixed point \( \tilde{x} \) of \( T \) which solves the variational inequality:

\[
\langle (A - \gamma \phi)\tilde{x}, J_q(\tilde{x} - z) \rangle \leq 0, \quad z \in F(T).
\]  

(2.3)

Proof. The definition of \( \{x_t\} \) is well definition. Indeed, from the definition of MKC, we can see MKC is also a nonexpansive mapping. Consider a mapping \( S_t \) on \( C \) defined by

\[
S_t x = t\gamma \phi(x) + (I - tA)Tx, \quad x \in C.
\]  

(2.4)
It is easy to see that \( S_t \) is a contraction. Indeed, by Lemma 2.8, we have

\[
\|S_t x - S_t y\| \leq t \|\phi(x) - \phi(y)\| + \|(I - tA)(Tx - Ty)\|
\leq t \|\phi(x) - \phi(y)\| + (1 - t\gamma)\|x - y\|
\leq t \|x - y\| + (1 - t\gamma)\|x - y\|
\leq [1 - t(\gamma - \gamma)]\|x - y\|.
\]

(2.5)

Hence, \( S_t \) has a unique fixed point, denoted by \( x_t \), which uniquely solves the fixed point equation

\[
x_t = t\gamma\phi(x_t) + (I - tA)Tx_t.
\]

(2.6)

We next show the uniqueness of a solution of the variational inequality (2.3). Suppose both \( \bar{x} \in F(T) \) and \( \tilde{x} \in F(T) \) are solutions to (2.3), not lost generality, we may assume there is a number \( \varepsilon \) such that \( \|\bar{x} - \tilde{x}\| \geq \varepsilon \). Then by Lemma 2.8, there is a number \( r \) such that \( \|\phi\bar{x} - \phi\tilde{x}\| \leq r\|\bar{x} - \tilde{x}\| \). From (2.3), we know

\[
\langle (A - \gamma\phi)\bar{x}, J_q(\bar{x} - \tilde{x}) \rangle \leq 0,
\]

(2.7)

\[
\langle (A - \gamma\phi)\tilde{x}, J_q(\bar{x} - \tilde{x}) \rangle \leq 0.
\]

Adding up (2.7) gets

\[
\langle (A - \gamma\phi)\bar{x} - (A - \gamma\phi)\tilde{x}, J_q(\bar{x} - \tilde{x}) \rangle \leq 0.
\]

(2.8)

Noticing that

\[
\langle (A - \gamma\phi)\bar{x} - (A - \gamma\phi)\tilde{x}, J_q(\bar{x} - \tilde{x}) \rangle = \langle A(\bar{x} - \tilde{x}), J_q(\bar{x} - \tilde{x}) \rangle - \gamma\langle \phi\bar{x} - \phi\tilde{x}, J_q(\bar{x} - \tilde{x}) \rangle
\]

\[
\geq \gamma\|\bar{x} - \tilde{x}\|^q - \gamma\|\phi\bar{x} - \phi\tilde{x}\|\|\bar{x} - \tilde{x}\|^{q-1}
\]

\[
\geq \gamma\|\bar{x} - \tilde{x}\|^q - \gamma r\|\bar{x} - \tilde{x}\|^q
\]

\[
\geq (\gamma - \gamma r)\|\bar{x} - \tilde{x}\|^q
\]

\[
\geq (\gamma - \gamma r)e^q
\]

(2.9)

Therefore \( \bar{x} = \tilde{x} \) and the uniqueness is proved. Below, we use \( \tilde{x} \) to denote the unique solution of (2.3).

We observe that \( \{x_t\} \) is bounded. Indeed, we may assume, with no loss of generality, \( t < \|A\|^{-1} \), for all \( p \in F(T) \), fixed \( \varepsilon_1 \), for each \( t \in (0, 1) \).

Case 1 (\( \|x_t - p\| < \varepsilon_1 \)). In this case, we can see easily that \( \{x_t\} \) is bounded.
Case 2 ($\|x_i - p\| \geq \epsilon_1$). In this case, by Lemmas 2.7 and 2.8, there is a number $r_1$ such that

$$\|\phi(x_i) - \phi(p)\| < r_1 \|x_i - p\|,$$

$$\|x_i - p\| = \|t\gamma \phi(x_i) + (I - tA)Tx_i - p\|$$

$$= \|t(\gamma \phi(x_i) - Ap) + (I - tA)(Tx_i - p)\|$$

$$\leq t\|\gamma \phi(x_i) - Ap\| + (1 - t\gamma)\|x_i - p\|$$

$$\leq t\|\gamma \phi(x_i) - \gamma \phi(p)\| + \|\gamma \phi(p) - Ap\| + (1 - t\gamma)\|x_i - p\|$$

$$\leq t\gamma r_1 \|x_i - p\| + t\|\gamma \phi(p) - Ap\| + (1 - t\gamma)\|x_i - p\|,$$

therefore, $\|x_i - p\| \leq \|\gamma \phi(p) - Ap\|/(\gamma - \gamma r_1)$. This implies the $\{x_i\}$ is bounded.

To prove that $x_i \to \bar{x}$ ($\bar{x} \in F(T)$) as $t \to 0$.

Since $\{x_i\}$ is bounded and $E$ is reflexive, there exists a subsequence $\{x_{i_n}\}$ of $\{x_i\}$ such that $x_{i_n} \to x^*$. By $x_i - Tx_i = t(\gamma \phi(x_i) - ATx_i)$. We have $x_{i_n} - T x_{i_n} \to 0$, as $t_n \to 0$. Since $E$ satisfies Opial’s condition, it follows from Lemma 2.3 that $x^* \in F(T)$. We claim

$$\|x_{i_n} - x^*\| \to 0.$$  \hspace{1cm} (2.11)

By contradiction, there is a number $\epsilon_0$ and a subsequence $\{x_{i_n}\}$ of $\{x_i\}$ such that $\|x_{i_n} - x^*\| \geq \epsilon_0$. From Lemma 2.8, there is a number $r_{\epsilon_0} > 0$ such that $\|\phi(x_{i_n}) - \phi(x^*)\| \leq r_{\epsilon_0} \|x_{i_n} - x^*\|$, we write

$$x_{i_n} - x^* = t_m(\gamma \phi(x_{i_n}) - Ax^*) + (I - t_mA)(Tx_{i_n} - x^*),$$  \hspace{1cm} (2.12)

to derive that

$$\|x_{i_n} - x^*\|^q = t_m(\gamma \phi(x_{i_n}) - Ax^*), J_q(x_{i_n} - x^*) + (I - t_mA)(Tx_{i_n} - x^*), J_q(x_{i_n} - x^*)$$

$$\leq t_m(\gamma \phi(x_{i_n}) - Ax^*), J_q(x_{i_n} - x^*) + (1 - t_m\gamma)\|x_{i_n} - x^*\|^q.$$  \hspace{1cm} (2.13)

It follows that

$$\|x_{i_n} - x^*\|^q \leq \frac{1}{\gamma} \langle \gamma \phi(x_{i_n}) - Ax^*, J_q(x_{i_n} - x^*) \rangle$$

$$= \frac{1}{\gamma} \left[ \langle \gamma \phi(x_{i_n}) - \gamma \phi(x^*), J_q(x_{i_n} - x^*) \rangle + \langle \gamma \phi(x^*) - Ax^*, J_q(x_{i_n} - x^*) \rangle \right]$$

$$\leq \frac{1}{\gamma} \left[ \gamma r_{\epsilon_0} \|x_{i_n} - x^*\|^q + \langle \gamma \phi(x^*) - Ax^*, J_q(x_{i_n} - x^*) \rangle \right].$$  \hspace{1cm} (2.14)

Therefore,

$$\|x_{i_n} - x^*\|^q \leq \frac{\langle \gamma \phi(x^*) - Ax^*, J_q(x_{i_n} - x^*) \rangle}{\gamma - \gamma r_{\epsilon_0}}.$$  \hspace{1cm} (2.15)
Using that the duality map \( J_q \) is single valued and weakly sequentially continuous from \( E \) to \( E^* \), by (2.15), we get that \( x_{n_m} \to x^* \). It is a contradiction. Hence, we have \( x_{n} \to x^* \).

We next prove that \( x^* \) solves the variational inequality (2.3). Since

\[
x_i = t\gamma \phi(x_i) + (I - tA)Tx_i, \tag{2.16}
\]

we derive that

\[
(A - \gamma \phi)x_i = -\frac{1}{t}(I - tA)(I - T)x_i. \tag{2.17}
\]

Notice

\[
\langle (I - T)x_i - (I - T)z, J_q(x_i - z) \rangle \geq \|x_i - z\|^q - \|Tx_i - Tz\|\|x_i - z\|^{q-1} \\
\geq \|x_i - z\|^q - \|x_i - z\|^q \\
= 0.
\]

It follows that, for \( z \in F(T) \),

\[
\langle (A - \gamma \phi)x_i, J_q(x_i - z) \rangle = -\frac{1}{t} \langle (I - tA)(I - T)x_i, J_q(x_i - z) \rangle \\
= -\frac{1}{t} \langle (I - T)x_i - (I - T)z, J_q(x_i - z) \rangle + \langle A(I - T)x_i, J_q(x_i - z) \rangle \\
\leq \langle A(I - T)x_i, J_q(x_i - z) \rangle.
\]

Now replacing \( t \) in (2.19) with \( t_n \) and letting \( n \to \infty \), noticing \( (I - T)x_{n_m} \to (I - T)x^* = 0 \) for \( x^* \in F(T) \), we obtain \( \langle (A - \gamma \phi)x^*, J_q(x^* - z) \rangle \leq 0 \). That is, \( x^* \in F(T) \) is a solution of (2.3); Hence \( \bar{x} = x^* \) by uniqueness. In a summary, we have shown that each cluster point of \( \{x_i\} \) (at \( t \to 0 \)) equals \( \bar{x} \), therefore, \( x_i \to \bar{x} \) as \( t \to 0 \).

\[\Box\]

**Lemma 2.10** (see, e.g., Mitrović [19, page 63]). Let \( q > 1 \). Then the following inequality holds:

\[
ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{q/(q-1)}, \tag{2.20}
\]

for arbitrary positive real numbers \( a, b \).

**Lemma 2.11.** Let \( E \) be a \( q \)-uniformly smooth Banach space which admits a weakly sequentially continuous dual duality mapping \( J_q \) from \( E \) to \( E^* \) and \( C \) be a nonempty convex subset of \( E \). Assume that \( T_i : C \to E \) is a countable family of \( \lambda_i \)-strict pseudocontraction for some \( 0 < \lambda_i < 1 \) and

\[
\inf\{\lambda_i : i \in \mathbb{N}\} > 0 \text{ such that } F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset.
\]

Assume that \( \{\eta_i\}_{i=1}^{\infty} \) is a positive sequence such that \( \sum_{i=1}^{\infty} \eta_i = 1 \). Then \( \sum_{i=1}^{\infty} \eta_i T_i : C \to E \) is a \( \lambda \)-strict pseudocontraction with \( \lambda = \inf\{\lambda_i : i \in \mathbb{N}\} \) and \( F(\sum_{i=1}^{\infty} \eta_i T_i) = F \).
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Proof. Let

\[ G_n x = \eta_1 T_1 x + \eta_2 T_2 x + \cdots + \eta_n T_n x \]  

(2.21)

and \( \sum_{i=1}^n \eta_i = 1 \). Then, \( G_n : C \to E \) is a \( \lambda_i \)-strict pseudocontraction with \( \lambda = \min\{\lambda_i : 1 \leq i \leq n\} \). Indeed, we can firstly see the case of \( n = 2 \).

\[ \langle (I - G_2) x - (I - G_2) y, J q (x - y) \rangle \]
\[ = \langle \eta_1 (I - T_1) x + \eta_2 (I - T_2) x - \eta_1 (I - T_1) y - \eta_2 (I - T_2) y, J q (x - y) \rangle \]
\[ = \eta_1 \langle (I - T_1) x - (I - T_1) y, J q (x - y) \rangle + \eta_2 \langle (I - T_2) x - (I - T_2) y, J q (x - y) \rangle \]
\[ \geq \eta_1 \lambda_1 \| (I - T_1) x - (I - T_1) y \| ^2 + \eta_2 \lambda_2 \| (I - T_2) x - (I - T_2) y \| ^2 \]
\[ \geq \lambda [ \eta_1 \| (I - T_1) x - (I - T_1) y \| ^2 + \eta_2 \| (I - T_2) x - (I - T_2) y \| ^2 ] \]
\[ \geq \lambda \| (I - G_2) x - (I - G_2) y \| ^2, \]

which shows that \( G_2 : C \to E \) is a \( \lambda \)-strict pseudocontraction with \( \lambda = \min\{\lambda_i : i = 1, 2\} \). By the same way, our proof method easily carries over to the general finite case.

Next, we prove the infinite case. From the definition of \( \lambda \)-strict pseudocontraction, we know

\[ \langle (I - T_n) x - (I - T_n) y, J q (x - y) \rangle \geq \lambda \| (I - T_n) x - (I - T_n) y \| ^q. \]  

(2.23)

Hence, we can get

\[ \| (I - T_n) x - (I - T_n) y \| \leq \left( \frac{1}{\lambda} \right)^{1/(q-1)} \| x - y \|. \]  

(2.24)

Taking \( p \in F(T_n) \), from (2.24), we have

\[ \| (I - T_n) x \| = \| (I - T_n) x - (I - T_n) p \| \leq \left( \frac{1}{\lambda} \right)^{1/(q-1)} \| x - p \|. \]  

(2.25)

Consequently, for all \( x \in E \), if \( F = \bigcap_{n=1}^\infty F(T_i) \neq \emptyset \), \( \eta_i > 0 \) (\( i \in \mathbb{N} \)) and \( \sum_{i=1}^\infty \eta_i = 1 \), then \( \sum_{i=1}^\infty \eta_i T_i \) strongly converges. Let

\[ T x = \sum_{i=1}^\infty \eta_i T_i x, \]  

(2.26)

we have

\[ T x = \sum_{i=1}^\infty \eta_i T_i x = \lim_{n \to \infty} \sum_{i=1}^n \eta_i T_i x = \lim_{n \to \infty} \frac{1}{\sum_{i=1}^n \eta_i} \sum_{i=1}^n \eta_i T_i x. \]  

(2.27)
Hence,
\[
\langle (I-T)x - (I-T)y, J_q(x-y) \rangle \\
= \lim_{n \to \infty} \left\langle \left( I - \frac{1}{\sum_{i=1}^{n} \eta_i T_i} \right) x + \left( I - \frac{1}{\sum_{i=1}^{n} \eta_i T_i} \right) y, J_q(x-y) \right\rangle \\
= \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \eta_i} \sum_{i=1}^{n} \eta_i \langle (I-T_i)x - (I-T_i)y, J_q(x-y) \rangle \\
\geq \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \eta_i} \sum_{i=1}^{n} \eta_i \lambda \| (I-T_i)x - (I-T_i)y \|^q \\
\geq \lambda \lim_{n \to \infty} \left\| \left( I - \frac{1}{\sum_{i=1}^{n} \eta_i} \sum_{i=1}^{n} \eta_i T_i \right) x - \left( I - \frac{1}{\sum_{i=1}^{n} \eta_i} \sum_{i=1}^{n} \eta_i T_i \right) y \right\|^q \\
= \lambda \| (I-T)x - (I-T)y \|^q.
\] (2.28)

So, we get $T$ is $\lambda$-strict pseudocontraction.

Finally, we show $F(\sum_{i=1}^{\infty} \eta_i T_i) = F$. Suppose that $x = \sum_{i=1}^{\infty} \eta_i T_i x$, it is sufficient to show that $x \in F$. Indeed, for $p \in F$, we have
\[
\|x - p\|^q = \langle x - p, J_q(x-p) \rangle \\
= \sum_{i=1}^{\infty} \eta_i \langle T_i x - p, J_q(x-p) \rangle \\
= \sum_{i=1}^{\infty} \eta_i \langle T_i x - p, J_q(x-p) \rangle \\
\leq \|x - p\|^q - \lambda \sum_{i=1}^{\infty} \eta_i \|x - T_i x\|^q,
\]
where $\lambda = \inf \{ \lambda_i : i \in \mathbb{N} \}$. Hence, $x = T_i x$ for each $i \in \mathbb{N}$, this means that $x \in F$. \hfill \square

3. Main Results

**Lemma 3.1.** Let $E$ be a real $q$-uniformly smooth, strictly convex Banach space and $C$ be a closed convex subset of $E$ such that $C \subseteq C \subseteq C$. Let $C$ be also a sunny nonexpansive retraction of $E$. Let $\phi : C \to C$ be a MKC. Let $A : C \to C$ be a strongly positive linear bounded operator with the coefficient $\gamma > 0$ such that $0 < \gamma < \gamma$ and $T_i : C \to E$ be $\lambda_i$-strictly pseudo-contractive non-self-mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\lambda = \inf \{ \lambda_i : i \in \mathbb{N} \} > 0$. Let $\{x_n\}$ be a sequence of $C$ generated by (1.12) with the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$, assume for each $n$, $\{\eta_i^{(n)}\}$ be an infinity sequence of positive number such that $\sum_{i=1}^{\infty} \eta_i^{(n)} = 1$ for all $n$ and $\eta_i^{(n)} > 0$. The following control conditions are satisfied

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0$, 
(ii) $1 - \alpha \leq 1 - \beta_n \leq \mu, \mu = \min \{ 1, \{q \lambda/C_q\}^{1/(q-1)} \}$ for some $\alpha \in (0,1)$ and for all $n \geq 0$, 

(iii) \( \lim_{n \to \infty} (\beta_{n+1} - \beta_n) = 0, \lim_{n \to \infty} \sum_{i=1}^{\infty} |\eta_i^{n+1} - \eta_i^n| = 0, \)
(iv) \( 0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1. \)

Then, \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \)

Proof. Write, for each \( n \geq 0, \) \( B_n = \sum_{i=1}^{\infty} \eta_i^{(n)} T_i. \) By Lemma 2.11, each \( B_n \) is a \( \lambda \)-strict pseudocontraction on \( C \) and \( F(B_n) = F \) for all \( n \) and the algorithm (1.12) can be rewritten as

\[
x_1 = x \in C,
\]
\[
y_n = P_C [ \beta_n x_n + (1 - \beta_n) B_n x_n ],
\]
\[
x_{n+1} = \alpha_n \gamma \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n A) y_n, \quad n \geq 1.
\]

The rest of the proof will now be split into two parts.

**Step 1.** First, we show that sequences \( \{x_n\} \) and \( \{y_n\} \) are bounded. Define a mapping

\[
L_n x := P_C [ \beta_n x + (1 - \beta_n) B_n x ].
\]

Then, from the control condition (ii), Lemmas 2.5 and 2.6, we obtain \( L_n : C \to C \) is nonexpansive. Taking a point \( p \in F \), by Lemma 2.4, we can get \( L_n p = p. \) Hence, we have

\[
\|y_n - p\| = \|L_n x_n - p\| \leq \|x_n - p\|.
\]

From definition of MKC and Lemma 2.8, for each \( \varepsilon > 0 \) there is a number \( r_\varepsilon \in (0, 1), \) if \( \|x_n - z\| < \varepsilon \) then \( \|\phi(x_n) - \phi(z)\| < \varepsilon; \) if \( \|x_n - z\| \geq \varepsilon \) then \( \|\phi(x_n) - \phi(z)\| \leq r_\varepsilon \|x_n - z\|. \) It follow (3.1)

\[
\|x_{n+1} - p\| = \|\alpha_n \gamma \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n A) y_n - p\|
\]
\[
= \|\alpha_n (\gamma \phi(x_n) - A p) + \gamma_n (x_n - p) + ((1 - \gamma_n) I - \alpha_n A) y_n - p\|
\]
\[
\leq (1 - \gamma_n - \alpha_n \gamma) \|x_n - p\| + \gamma_n \|x_n - p\| + \alpha_n \|\gamma \phi(x_n) - A p\|
\]
\[
\leq (1 - \alpha_n \gamma) \|x_n - p\| + \gamma_n \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|
\]
\[
= \max \{ (1 - \alpha_n \gamma) \|x_n - p\| + \alpha_n \gamma r_\varepsilon \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|,
\]
\[
(1 - \alpha_n \gamma) \|x_n - p\| + \alpha_n \gamma \varepsilon + \alpha_n \|\gamma \phi(p) - A p\|
\]
\[
+ \alpha_n \gamma r_\varepsilon \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|
\]
\[
= \max \{ [1 - (\alpha_n \gamma + \alpha_n\gamma r_\varepsilon)] \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|,
\]
\[
(1 - \alpha_n \gamma) \|x_n - p\| + \alpha_n \gamma \varepsilon + \alpha_n \|\gamma \phi(p) - A p\|
\]
\[
+ \alpha_n \gamma r_\varepsilon \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|
\}
\]

\[
= \max \{ [1 - (\alpha_n \gamma + \alpha_n\gamma r_\varepsilon)] \|x_n - p\| + \alpha_n \|\gamma \phi(p) - A p\|,
\]
\[
(1 - \alpha_n \gamma) \|x_n - p\| + \alpha_n \gamma \varepsilon + \alpha_n \|\gamma \phi(p) - A p\|
\}
\]
By induction, we have
\[
\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|\gamma \phi(p) - Ap\|}{\gamma} + \frac{\|\gamma \phi(p) - Ap\|}{\gamma} \right\}, \quad n \geq 1, \tag{3.5}
\]
which gives that the sequence \(\{x_n\}\) is bounded, so are \(\{y_n\}\) and \(\{L_n x_n\}\).

Step 2. In this part, we shall claim that \(\|x_{n+1} - x_n\| \to 0\), as \(n \to \infty\). From (3.1), we get
\[
x_{n+1} = \alpha_n \gamma \phi(x_n) + \gamma_n x_n + \left[ (1 - \gamma_n) I - \alpha_n A \right] L_n x_n, \tag{3.6}
\]
Define
\[
x_{n+1} = (1 - \gamma_n) I_n + \gamma_n x_n, \quad \forall n \geq 0, \tag{3.7}
\]
where
\[
l_n = \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n}. \tag{3.8}
\]
It follows that
\[
l_{n+1} - l_n = \frac{\alpha_n [\gamma \phi(x_{n+1}) - AL_n x_{n+1}]}{1 - \gamma_{n+1}} - \frac{\alpha_n [\gamma \phi(x_n) - AL_n x_n]}{1 - \gamma_n} + l_{n+1} x_{n+1} - L_n x_n, \tag{3.9}
\]
which yields that
\[
\|l_{n+1} - l_n\| \leq \frac{\alpha_n \|\gamma \phi(x_{n+1}) - AL_n x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - AL_n x_n\|}{1 - \gamma_n} + \|L_{n+1} x_{n+1} - L_n x_n\|
\]
\[
\leq \frac{\alpha_n \|\gamma \phi(x_{n+1}) - AL_n x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - AL_n x_n\|}{1 - \gamma_n} + \|L_{n+1} x_{n+1} - L_n x_n\|
\]
\[
+ \|L_{n+1} x_n - L_n x_n\|
\]
\[
\leq \frac{\alpha_n \|\gamma \phi(x_{n+1}) - AL_n x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - AL_n x_n\|}{1 - \gamma_n} + \|x_{n+1} - x_n\|
\]
\[
+ \|L_{n+1} x_n - L_n x_n\|. \tag{3.10}
\]
Next, we estimate $\|L_{n+1}x_n - L_n x_n\|$. Notice that

$$
\|L_{n+1}x_n - L_n x_n\| = \|P_C[\beta_{n+1}x_n + (1 - \beta_{n+1})B_{n+1}x_n] - P_C[\beta_nx_n + (1 - \beta_n)B_nx_n]\|
\leq \|\beta_{n+1}x_n + (1 - \beta_{n+1})B_{n+1}x_n - [\beta_nx_n + (1 - \beta_n)B_nx_n]\|
\leq |\beta_{n+1} - \beta_n|\|x_n - B_{n+1}x_n\| + (1 - \beta_n)\|B_{n+1}x_n - B_n x_n\|
\leq |\beta_{n+1} - \beta_n|\|x_n - B_{n+1}x_n\| + (1 - \beta_n)\sum_{i=1}^{\infty} |\eta^{(n+1)}_i - \eta^{(n)}_i|\|T_i x_n\|.
$$

(3.11)

Substituting (3.11) into (3.10), we have

$$
\|I_{n+1} - I_n\| \leq \frac{\alpha_n}{1 - \gamma_{n+1}}\|\gamma \phi(x_{n+1}) - AL_{n+1}x_{n+1}\| + \frac{\alpha_n}{1 - \gamma_n}\|\gamma \phi(x_n) - AL_n x_n\|
+ |\beta_{n+1} - \beta_n|\|x_n - B_{n+1}x_n\| + (1 - \beta_n)\sum_{i=1}^{\infty} |\eta^{(n+1)}_i - \eta^{(n)}_i|\|T_i x_n\|.
$$

(3.12)

Hence, we have

$$
\|I_{n+1} - I_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_n}{1 - \gamma_{n+1}}\|\gamma \phi(x_{n+1}) - AL_{n+1}x_{n+1}\| + \frac{\alpha_n}{1 - \gamma_n}\|\gamma \phi(x_n) - AL_n x_n\|
+ \|x_n - B_{n+1}x_n\| |\beta_{n+1} - \beta_n| + (1 - \beta_n)\sum_{i=1}^{\infty} |\eta^{(n+1)}_i - \eta^{(n)}_i|\|T_i x_n\|.
$$

(3.13)

Observing conditions (i), (iii), (iv), and the boundedness of \(\{x_n\}, \{y_n\}, \{f(x_n)\}, \{T_n x_n\}, \{T_n y_n\}\) it follows that

$$
\limsup_{n \to \infty} \{(I_{n+1} - I_n) - (x_{n+1} - x_n)\} \leq 0.
$$

(3.14)

Thus by Lemma 2.1, we have $\lim_{n \to \infty} \|I_n - x_n\| = 0$.

From (3.7), we have

$$
x_{n+1} - x_n = (1 - \gamma_n)(I_n - x_n).
$$

(3.15)

Therefore,

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
$$

(3.16)

**Theorem 3.2.** Let $E$ be a real $q$-uniformly smooth, strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J_q$ from $E$ to $E^*$ and $C$ be a closed convex subset of $E$ which be also a sunny nonexpansive retraction of $E$ such that $C \pm C \subset C$. Let $\phi : C \to C$ be
a MKC. Let $A : C \rightarrow C$ be a strongly positive linear bounded operator with the coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}$ and $T_i : C \rightarrow E$ be $\lambda_i$-strictly pseudo-contractive non-self-mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\lambda = \inf \{ \lambda_i : i \in \mathbb{N} \} > 0$. Let $\{x_n\}$ be a sequence of $C$ generated by (1.12) with the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$, assume for each $n$, $\sum_{i=1}^{\infty} \eta_i^{(n)} = 1$ for all $n$ and $\eta_i^{(n)} > 0$ for all $i \in \mathbb{N}$. They satisfy the conditions (ii), (iii), (iv) of Lemma 3.1 and (v) $\lim_{n \rightarrow \infty} \beta_n = \alpha$, $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} |\eta_i^{(n)} - \eta_i| = 0$ and $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\{x_n\}$ converges strongly to $\bar{x} \in F$, which also solves the following variational inequality

$$\langle \gamma \phi(\bar{x}) - A\bar{x}, J_q(p - \bar{x}) \rangle \leq 0, \quad \forall p \in F. \quad (3.17)$$

**Proof.** From (3.1), we obtain

$$\|L_n x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - L_n x_n\|
= \|x_n - x_{n+1}\| + \|\alpha_n \gamma \phi(x_n) + \gamma_n (x_n - L_n x_n) - \alpha_n A L_n x_n\|
\leq \|x_n - x_{n+1}\| + \alpha_n (\|\gamma \phi(x_n)\| + \|A L_n x_n\|) + \gamma_n \|x_n - L_n x_n\|. \quad (3.18)$$

So $\|L_n x_n - x_n\| \leq 1/(1 - \gamma_n)(\|x_n - x_{n+1}\| + \alpha_n (\|\gamma \phi(x_n)\| + \|A L_n x_n\|))$, which together with the condition (i), (iv) and Lemma 3.1 implies

$$\lim_{n \rightarrow -\infty} \|L_n x_n - x_n\| = 0. \quad (3.19)$$

Define $B = \sum_{i=1}^{\infty} \eta_i T_i$, then $B : C \rightarrow E$ is a $\lambda$-strict pseudo-contraction such that $F(B) = \bigcap_{i=1}^{\infty} F(T_i) = F$ by Lemma 2.11, furthermore $B_n x \rightarrow Bx$ as $n \rightarrow \infty$ for all $x \in C$. Defines $T : C \rightarrow E$ by

$$Tx = \alpha x + (1 - \alpha)Bx. \quad (3.20)$$

Then, $T$ is nonexpansive with $F(T) = F(B)$ by Lemma 2.5. It follows from Lemma 2.4 that $F(P_C T) = F(T) = F$. Notice that

$$\|P_C Tx_n - x_n\| \leq \|x_n - L_n x_n\| + \|L_n x_n - P_C Tx_n\|
\leq \|x_n - L_n x_n\| + \|\beta_n x_n + (1 - \beta_n)B_n x_n - [\alpha x_n + (1 - \alpha)Bx_n]\|
\leq \|x_n - L_n x_n\| + \|\beta_n x_n + (1 - \beta_n)(x_n - B_n x_n) + (1 - \alpha)(B_n x_n - B x_n)\|
\leq \|x_n - L_n x_n\| + (\beta_n - \alpha) \|x_n - B_n x_n\| + (1 - \alpha) \|B_n x_n - B x_n\| \quad (3.21)$$

which combines with (3.19) yielding that

$$\lim_{n \rightarrow -\infty} \|P_C Tx_n - x_n\| = 0. \quad (3.22)$$
Next, we show that
\[ \limsup_{n \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x}) \rangle \leq 0, \quad (3.23) \]

where \( \tilde{x} = \lim_{t \to 0} x_t \) with \( x_t \) being the fixed point of the contraction
\[ x \mapsto t \gamma \phi(x) + (1 - tA) PCTx. \quad (3.24) \]

To see this, we take a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that
\[ \limsup_{n \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x}) \rangle = \lim_{k \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_{n_k} - \tilde{x}) \rangle. \quad (3.25) \]

We may also assume that \( x_{n_k} \rightharpoonup q \). Note that \( q \in F(T) \) in virtue of Lemma 2.3 and (3.22). It follow from the Lemma 2.9 and \( J_q \) is weak weakly sequentially continuous duality mapping that
\[ \limsup_{n \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x}) \rangle = \lim_{k \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_{n_k} - \tilde{x}) \rangle \]
\[ = \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(q - \tilde{x}) \rangle \leq 0. \quad (3.26) \]

Hence, we have
\[ \limsup_{n \to \infty} \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x}) \rangle \leq 0. \quad (3.27) \]

Finally, We show \( \|x_n - \tilde{x}\| \to 0 \). By contradiction, there is a number \( \varepsilon_0 \) such that
\[ \limsup_{n \to \infty} \|x_n - \tilde{x}\| \geq \varepsilon_0. \quad (3.28) \]

Case 1. Fixed \( \varepsilon_1 (\varepsilon_1 < \varepsilon_0) \), if for some \( n \geq N \in \mathbb{N} \) such that \( \|x_n - \tilde{x}\| \geq \varepsilon_0 - \varepsilon_1 \), and for the other \( n \geq N \in \mathbb{N} \) such that \( \|x_n - \tilde{x}\| < \varepsilon_0 - \varepsilon_1 \).

Let
\[ M_n = \frac{q \langle \gamma \phi(\tilde{x}) - A\tilde{x}, J_q(x_{n+1} - \tilde{x}) \rangle}{(\varepsilon_0 - \varepsilon_1)^d}. \quad (3.29) \]
From (3.23), we know \( \limsup_{n \to \infty} M_n \leq 0 \). Hence, there is a number \( N \), when \( n > N \), we have \( M_n \leq \gamma - \gamma \). We extract a number \( n_0 \geq N \) satisfying \( \|x_{n_k} - \bar{x}\| < \varepsilon_0 - \varepsilon_1 \), then we estimate \( \|x_{n_k+1} - \bar{x}\| \).

\[
\|x_{n_k+1} - \bar{x}\|^q = \|\alpha_n y_\bar{\phi}(x_{n_k}) + y_n x_n + [(1 - y_n)I - \alpha_n A]y_{n_0} - \bar{x}\|^q \\
= \|[(1 - y_n)I - \alpha_n A](y_{n_0} - \bar{x}) + \alpha_n (\gamma\phi(x_{n_k}) - A\bar{x}) + y_n(x_{n_k} - \bar{x})\|^q \\
= \langle [(1 - y_n)I - \alpha_n A](y_{n_0} - \bar{x}) + \alpha_n (\gamma\phi(x_{n_k}) - A\bar{x}) + y_n(x_{n_k} - \bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \\
= \langle [(1 - y_n)I - \alpha_n A](y_{n_0} - \bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle + \alpha_n (\gamma\phi(x_{n_k}) - A\bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle + \langle y_n(x_{n_k} - \bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \\
\leq (1 - \alpha_n - \alpha_n - \gamma)(\|x_{n_0} - \bar{x}\||x_{n_k+1} - \bar{x}|^{q-1} + \|\gamma\phi(x_{n_k}) - \phi(\bar{x})\||x_{n_k+1} - \bar{x}|^{q-1} + \|y_n\|\|x_{n_k} - \bar{x}\||x_{n_k+1} - \bar{x}|^{q-1} \\
\leq [1 - \alpha_n(\bar{\gamma} - \gamma)](\varepsilon_0 - \varepsilon_1)|x_{n_k+1} - \bar{x}|^{q-1} + \alpha_n(\gamma\phi(\bar{x}) - A\bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \\
\leq \frac{1}{q}[1 - \alpha_n(\bar{\gamma} - \gamma)]^q(\varepsilon_0 - \varepsilon_1)^q + \frac{q - 1}{q}\|x_{n_k+1} - \bar{x}\|^q \\
+ \alpha_n(\gamma\phi(\bar{x}) - A\bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \text{ by Lemma 2.10,} \\
(3.30)
\]

which implies that

\[
\|x_{n_k+1} - \bar{x}\|^q < [1 - \alpha_n(\bar{\gamma} - \gamma)]^q(\varepsilon_0 - \varepsilon_1)^q + q\alpha_n(\gamma\phi(\bar{x}) - A\bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \\
< [1 - \alpha_n(\bar{\gamma} - \gamma)](\varepsilon_0 - \varepsilon_1)^q + q\alpha_n(\gamma\phi(\bar{x}) - A\bar{x}), J_q(x_{n_k+1} - \bar{x}) \rangle \\
= [1 - \alpha_n(\bar{\gamma} - \gamma - M_n)](\varepsilon_0 - \varepsilon_1)^q \\
\leq (\varepsilon_0 - \varepsilon_1)^q. \\
(3.31)
\]

Hence, we have

\[
\|x_{n_k+1} - \bar{x}\| < \varepsilon_0 - \varepsilon_1. \\
(3.32)
\]

In the same way, we can get

\[
\|x_n - \bar{x}\| < \varepsilon_0 - \varepsilon_1, \quad \forall n \geq n_0. \\
(3.33)
\]

It contradict the \( \limsup_{n \to \infty} \|x_n - \bar{x}\| \geq \varepsilon_0 \).
Case 2. Fixed $\varepsilon_1 (\varepsilon_1 < \varepsilon_0)$, if $\|x_n - \bar{x}\| \geq \varepsilon_0 - \varepsilon_1$ for all $n \geq N \in \mathbb{N}$, from Lemma 2.8, there is a number $r, (0 < r < 1)$ such that

$$\|\phi(x_n) - \phi(\bar{x})\| \leq r\|x_n - \bar{x}\|, \quad n \geq N. \tag{3.34}$$

It follow (3.1) that

$$\|x_{n+1} - \bar{x}\|^q = \|\alpha_n \gamma \phi(x_n) + \gamma_n x_n + \left[(1 - \gamma_n)I - \alpha_n A\right]y_n - \bar{x}\|^q$$

$$= \|\left[(1 - \gamma_n)I - \alpha_n A\right](y_n - \bar{x}) + \alpha_n(\gamma \phi(x_n) - A\bar{x}) + \gamma_n(x_n - \bar{x})\|^q$$

$$= \left\langle \left[(1 - \gamma_n)I - \alpha_n A\right](y_n - \bar{x}), J_q(x_{n+1} - \bar{x})\right\rangle + \langle \alpha_n(\gamma \phi(x_n) - A\bar{x}), J_q(x_{n+1} - \bar{x})\rangle$$

$$+ \langle \gamma_n(x_n - \bar{x}), J_q(x_{n+1} - \bar{x})\rangle$$

$$\leq (1 - \gamma_n - \alpha_n \gamma)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} + \alpha_n \gamma r\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1}$$

$$+ \alpha_n(\gamma \phi(\bar{x}) - A\bar{x}, J_q(x_{n+1} - \bar{x})) + \gamma_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1}$$

$$\leq (1 - \gamma_n(\gamma - \gamma r))\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\|^{q-1} + \alpha_n(\gamma \phi(\bar{x}) - A\bar{x}, J_q(x_{n+1} - \bar{x}))$$

$$\leq \left[1 - \gamma_n(\gamma - \gamma r)\right]\frac{1}{q}\|x_n - \bar{x}\|^q + \frac{q-1}{q}\|x_{n+1} - \bar{x}\|^q + \alpha_n\|\gamma \phi(\bar{x}) - A\bar{x}, J_q(x_{n+1} - \bar{x})\|$$

$$= [1 - \alpha_n(\gamma - \gamma r)]\|x_n - \bar{x}\|^{q-1} + \alpha_n\|\gamma \phi(\bar{x}) - A\bar{x}, J_q(x_{n+1} - \bar{x})\|$$

which implies that

$$\|x_{n+1} - \bar{x}\|^q \leq [1 - \alpha_n(\gamma - \gamma r)]\|x_n - \bar{x}\|^{q-1} + q\alpha_n\|\gamma \phi(\bar{x}) - A\bar{x}, J_q(x_{n+1} - \bar{x})\|. \tag{3.36}$$

Apply Lemma 2.2 to (3.36) to conclude $x_n \to \bar{x}$ as $n \to \infty$. It contradict the $\|x_n - \bar{x}\| \geq \varepsilon_0 - \varepsilon_1$. This completes the proof. \hfill \square

**Corollary 3.3.** Let $D$ be a closed convex subset of a Hilbert space $H$ such that $D \pm D \subset D$ and $f \in D$ with the coefficient $0 < 0 < 1$. Let $A : C \to C$ be a strongly positive bounded operator with the coefficient $\gamma > 0$ such that $0 < \gamma < \gamma$ and $T_i : C \to E$ be $\lambda_i$-strictly pseudo-contractive non-self-mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\lambda_i = \inf \{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of $C$ generated by (1.12) with the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $[0, 1]$, assume for each $n$, $\sum_{i=1}^{\infty} \eta_i^{(n)} = 1$ for all $n$ and $\eta_i^{(n)} > 0$ for all $i \in \mathbb{N}$. They satisfy the conditions (i), (ii), (iii), (iv) of Lemma 3.1 and (v)
\[ \lim_{n \to \infty} \beta_n = \alpha, \lim_{n \to \infty} \sum_{i=1}^{\infty} |\eta_i^n - \eta_i| = 0 \text{ and } \sum_{i=1}^{\infty} \eta_i = 1. \text{ Then } \{x_n\} \text{ converges strongly to } \bar{x} \in F, \] which also solves the following variational inequality

\[ \langle \gamma \Phi(\bar{x}) - A\bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in F. \] (3.37)

Remark 3.4. We conclude the paper with the following observations.

(i) Theorem 3.2 improve and extends Theorem 3.1 of Zhang and Su [17], Theorem 1 of Yao et al. [11], and Theorem 2.2 of Cai and Hu [12]. Corollary 3.3 also improve and extend Theorem 2.1 of Choia et al. [20], Theorem 2.1 of Jung [21], Theorem 2.1 of Qin et al. [22] and includes those results as special cases. Especially, Our results extends above results form contractions to more general Meir-Keeler contraction (MKC, for short). Our iterative scheme studied in present paper can be viewed as a refinement and modification of the iterative methods in [12, 13, 17, 22]. On the other hand, our iterative schemes concern an infinite countable family of \( \lambda_i\)-strict pseudocontractions mappings, in this respect, they can be viewed as an another improvement.

(ii) The advantage of the results in this paper is that less restrictions on the parameters \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\eta_i^n\} \) are imposed. Our results unify many recent results including the results in [12, 17, 22].

(iii) It is worth noting that we obtained two strong convergence results concerning an infinite countable family of \( \lambda_i\)-strict pseudocontractions mappings. Our result is new and the proofs are simple and different from those in [11, 12, 17, 19–25].

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References


