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Research Article

On Some Properties of Hyperconvex Spaces

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We are going to answer some open questions in the theory of hyperconvex metric spaces. We prove that in complete $\mathbb{R}$-trees hyperconvex hulls are uniquely determined. Next we show that hyperconvexity of subsets of normed spaces implies their convexity if and only if the space under consideration is strictly convex. Moreover, we prove a Krein-Milman type theorem for $\mathbb{R}$-trees. Finally, we discuss a general construction of certain complete metric spaces. We analyse its particular cases to investigate hyperconvexity via measures of noncompactness.

1. Introduction

It is hard to believe that although hyperconvex metric spaces have been investigated for more that fifty years, some basic questions in their theory still remain open (let us recall that hyperconvex metric spaces were introduced in [1] (see also [2]), but from formal point of view it has to be emphasized that the notion of hyperconvexity was investigated earlier by Aronszajn in his Ph.D. thesis [3] which was never published). The main purpose of this paper is to answer some of these questions.

Let us begin with the notion of hyperconvex hull which was introduced by Isbell in [4] (see Definition 2.7). This notion is more difficult to investigate than the classical notion of convex hull, since the former one is not uniquely determined (see Proposition 2.8). In Section 3 we are going to prove that in hyperconvex metric spaces with the unique metric segments property, hyperconvex hulls are uniquely determined. Let us recall that such hyperconvex spaces were characterized by Kirk (see [5]) as complete $\mathbb{R}$-trees (see Theorem 2.15). This led to a surprising application of the theory of hyperconvex spaces to graph theory (see [6]).

Another interesting question is about the relation between the notion of convexity and hyperconvexity (cf. Remark 4.1). In particular, it is inspired by the following Sine's remark [7, page 863], stated without a proof: "The term hyperconvex does have some unfortunate aspects. First, a hyperconvex subset of even $\mathbb{R}^2$ (with the l_∞ norm) need not be convex. Also convex sets can fail to be hyperconvex (but for this one must go to at least $\mathbb{R}^3$)." It turns out that all hyperconvex subsets of a given normed space are convex if and only if the space in question is strictly convex; this fact is proved in Section 4.

In Section 5 we turn our attention to the classical Krein-Milman theorem (see [8]). We prove that a bounded complete $\mathbb{R}$-tree is a convex hull of its extremal points (note that a similar result, but with the assumption of compactness, is proved in [9]). Hence, in particular, such a property holds for bounded hyperconvex metric spaces with unique metric segments.

Let us denote by α and β the Kuratowski and Hausdorff measures of noncompactness, respectively, (see [10, 11] for the definition and basic properties). It was noticed by Espínola (see [12]) that if a metric space is hyperconvex, then $\alpha(A) = 2\beta(A)$ for all its bounded subsets A. The question is about the inverse implication. More precisely, assume that $\alpha(A) = 2\beta(A)$ for every bounded subset of a given metric space X. Does this equality imply that X is hyperconvex? (Obviously, we mean nontrivial cases, i.e., we exclude spaces in which every bounded set is relatively compact.) In Sections 6 and 7 we introduce a few metric spaces which are not hyperconvex, but $\alpha(A) = 2\beta(A)$ for all their bounded subsets. Hence the answer to the above question is negative. Let us emphasize that the metrics considered in Sections 6 and 7 are extensions and generalizations of commonly known radial metric and river metric, which were proved in [13] to be hyperconvex.

Let us notice that in general it is not easy to provide explicit formulae which would allow to evaluate the measures of noncompactness in particular spaces. We are going to state such formulae for the metric spaces considered in Sections 6 and 7.

Let us emphasize that another motivation to consider those metrics comes from the real world. Let us consider an example of the transmission of phone signals, when one person (say, v_1) calls another (say, v_2), assuming there are two base transceiver stations (say, A and B). We may have two cases. If v_1 and v_2 are in the range of one of the BTS's, say A, then the signal is first transmitted from v_1 to A and then from A to v_2—even if v_1 and v_2 are "close" to each other. If v_1 and v_2 are located in the ranges of A and B, respectively, then the signal is transmitted from v_1 to A, then from A to B and finally from B to v_2. Hence we have the metric considered in Definition 7.4.

In Section 8 we provide a general scheme to construct metrics similar to these of Sections 6 and 7. This scheme is a generalization of a construction from [14].

For completeness, in Section 2 we collect some basic definitions and facts used in the sequel.

2. Preliminaries

In what follows we will denote the Euclidean metric on $\mathbb{R}^n$ by ρ and a "maximum" norm on any suitable space by $\|\cdot\|_\infty$.

Let us begin with some classical definitions and facts.

Definition 2.1. Let (X, d) be a metric space. We call a set $S \subset X$ a *metric segment* (*joining the points* $p, q \in X$) if there exists an isometric embedding $i : [0, d(p, q)] \to X$ such that $i(0) = p$ and $i(d(p, q)) = q$.

Definition 2.2 (see [1, page 410, Definition 1]). We call a metric space *(X, d) hyperconvex*, if any family of closed balls $\{\overline{B}(x_i, r_i)\}_{i\in I}$ with centers at x_i's and radii of r_i's, respectively, such that $d(x_i, x_j) \leq r_i + r_j$ for any $i, j \in I$ has a nonempty intersection.

Hyperconvex spaces possess—among others—the following properties.

Proposition 2.3 (see [1, page 417, Theorem 1′]). *A hyperconvex space is complete.*

Proposition 2.4 (see [1, page 423, Theorem 9]). *A nonexpansive retract (i.e., a retract by a nonexpansive retraction) of a hyperconvex space is hyperconvex.*

Proposition 2.5 (see [1, page 422, Corollary 4]). *Each hyperconvex metric space is an absolute nonexpansive retract, that is, it is a nonexpansive retract of any metric space it is isometrically embedded in. In particular, hyperconvex spaces are absolute retracts.*

The following theorem gives a characterization of hyperconvex real Banach spaces.

Theorem 2.6 (Nachbin-Kelley, see [15, 16]). *A real Banach space is hyperconvex if and only if it is isometrically isomorphic to some space $C_{\mathbb{R}}(K)$ of all real continuous functions on a Hausdorff, compact and extremally disconnected topological space K with the* "sup" *norm.*

Now let us state the definition of a hyperconvex hull. We will not need the general version of this notion, investigated by Isbell in [4]; instead, the notion of a hyperconvex hull of a subset of a hyperconvex space will suffice for our considerations.

Definition 2.7 (see, e.g., [17, page 408]). Let $A \subset H$ be a nonempty subset of a hyperconvex space H. We call $B \subset H$ a *hyperconvex hull of* A (*in* H) if $A \subset B$, the set B is hyperconvex (as a metric subspace) and there exists no hyperconvex $B' \subset H$ such that $A \subset B' \subsetneq B$.

A hyperconvex hull always exists, but needs not to be unique. It is, however, unique up to an isometry. To be more precise, the following holds.

Proposition 2.8 (cf. [17, page 408, Proposition 5.6]). *Each nonempty subset of a hyperconvex metric space possesses a hyperconvex hull. If (X, d_X) and (Y, d_Y) are hyperconvex spaces, $A_X \subset X$, $A_Y \subset Y$ are isometric and $i : A_X \to A_Y$ is an isometry, then for any hyperconvex hulls $H_X \subset X$, $H_Y \subset Y$ of A_X and A_Y, respectively, the isometry i extends to an isometry $\tilde{\imath} : H_X \to H_Y$.*

In what follows, we will also need the definitions of total and strict convexity.

Definition 2.9 (see, e.g. [1, page 407] and [18, page 6, Definition 2.1]). A metric space (X, d) is called *totally convex* if for any two points $p, q \in X$ and for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$ there exists a point $r \in X$ satisfying the equalities $d(p, r) = \alpha d(p, q)$ and $d(r, q) = \beta d(p, q)$. If this point is unique for all possible combinations of p, q, α, β, we call the *space X strictly convex* and denote this point by $\alpha p + \beta q$.

Remark 2.10 (see [1, page 410]). A hyperconvex space is totally convex.

Remark 2.11 (see, e.g., [18, page 7]). For normed spaces, the above definition of strict convexity (Definition 2.9) coincides with the usual one.

Proposition 2.12 (see, e.g., [18, page 7]). *In a strictly convex metric space, intersection of any family of totally convex subsets is itself totally convex.*

The above proposition lets us define the notion of a *convex hull* in any strictly convex metric space in a natural way.

Definition 2.13. Let A be a nonempty subset of a strictly convex metric space X. The *convex hull of A (in X)* is the set

$$\mathrm{conv}_X A := \bigcap \{C \subset X \mid A \subset C \text{ and the subspace } C \text{ is totally convex}\}. \tag{2.1}$$

When the underlying space X is obvious from the context, we will usually write $\mathrm{conv}\, A$ instead of $\mathrm{conv}_X A$.

Now, let us recall the definition of an $\mathbb{R}$-tree.

Definition 2.14 (see, e.g., [5, page 68, Definition 1.2]). A metric space (T, d) is called an $\mathbb{R}$-*tree*, if the following conditions are satisfied:

(1) any two points $p, q \in T$ are joined by a unique metric segment (denoted by $[p, q]_d$);

(2) if $p, q, r \in T$ and $[p, q]_d \cap [q, r]_d = \{q\}$, then $[p, q]_d \cup [q, r]_d = [p, r]_d$;

(3) for any $p, q, r \in T$ there exists $s \in T$ such that $[p, q]_d \cap [p, r]_d = [p, s]_d$.

(Let us note that (3) follows from (1); it is, however, useful to have it among the basic properties of $\mathbb{R}$-trees.) We will also use the notation $(p, q)_d := [p, q]_d \setminus \{p, q\}$ for an *open metric segment joining p and q* and $(p, q]_d := [p, q]_d \setminus \{p\}$ for a *left-open* one.

Theorem 2.15 (see [5, Theorem 3.2]). *For a metric space X the following conditions are equivalent:*

(1) *X is a complete $\mathbb{R}$-tree;*

(2) *X is hyperconvex and any two points in X are joined by a unique metric segment.*

In what follows, we will also use the classical notions of Chebyshev subset of a metric space, a metric projection onto such a set C (which we will denote by P_C), Kuratowski and Hausdorff measures of noncompactness (which we will denote by α and β, resp.), and the radial and river metrics (which we will denote by d_r and d_{ri}, resp.). The reader may find the relevant definitions, for instance, in the papers [11, 19, 20].

3. $\mathbb{R}$-Trees

Let us begin this section with the following three simple propositions, which will enable us to characterize $\mathbb{R}$-trees as exactly these hyperconvex spaces in which hyperconvex hulls are unique.

Proposition 3.1. *A hyperconvex hull of a two-point subset $\{p, q\}$ of a hyperconvex metric space is a metric segment joining p and q.*

Proof. It is enough to consider $\{p, q\}$ as a subset of $\mathbb{R}$ and apply the uniqueness (up to isometry) of hyperconvex hulls (Proposition 2.8). □

Proposition 3.2. *$\mathbb{R}$-trees are strictly convex.*

Proof. Let (T, d) be an $\mathbb{R}$-tree. Assume that $x, y \in T$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $z_1, z_2 \in T$, $z_1 \neq z_2$ and $d(x, z_i) = \alpha d(x, y), d(z_i, y) = \beta d(x, y)$ for $i \in \{1, 2\}$. Then $[x, z_1]_d \neq [x, z_2]_d$, $[z_1, y]_d \neq [z_2, y]_d$. But we have $[x, z_i]_d \cap [z_i, y]_d = \{z_i\}$ for $i \in \{1, 2\}$ and therefore $[x, y]_d = [x, z_1]_d \cup [z_1, y]_d \neq [x, z_2]_d \cup [z_2, y]_d = [x, y]_d$, which is a contradiction. □

Proposition 3.3. *For a subset A of an $\mathbb{R}$-tree, the following conditions are equivalent:*

(1) *A is hyperconvex;*

(2) *A is closed and totally convex.*

Proof. For (1)⇒(2), it is enough to use Proposition 2.3 and Remark 2.10. On the other hand, if a subset C of an $\mathbb{R}$-tree T is closed and totally convex, it is a complete sub-$\mathbb{R}$-tree of T. Indeed, it is enough to show that for each $p, q \in C$, the metric segment $[p, q]_d \subset C$. But in view of the strict convexity of T, we have $[p, q]_d = \{\alpha p + \beta q \mid \alpha, \beta \geq 0, \alpha + \beta = 1\} \subset C$. Now, in view of Theorem 2.15, C is hyperconvex. □

A natural question to ask is: in which hyperconvex metric spaces the hyperconvex hulls are unique? The following theorem answers this question.

Theorem 3.4. *Let (H, d) be a hyperconvex metric space. The following conditions are equivalent:*

(1) *for each $A \subset H$ there exists exactly one hyperconvex hull of A in H;*

(2) *H is an $\mathbb{R}$-tree.*

Proof. Necessity follows easily from Proposition 3.1 and Theorem 2.15. *Sufficiency*. Let A be a subset of an $\mathbb{R}$-tree H. Notice that $\Sigma := \{B \subset H \mid A \subset B, B \text{ hyperconvex}\} = \{B \subset H \mid A \subset B,\ B \text{ closed and totally convex}\}$. Using Propositions 3.2, 2.12 and 3.3, we arrive at the conclusion that $\bigcap \Sigma$ is the hyperconvex hull of A in H. □

4. Normed Spaces

In the first part of this section we will give an answer to the following question: In which spaces closed and convex subsets are hyperconvex?

Remark 4.1. Note that the question whether all closed and convex subsets of some normed space are hyperconvex makes sense only in spaces which are themselves hyperconvex, so we will now restrict our attention to such spaces.

Theorem 4.2 (see [21, page 474, Theorem 1]). *If E is a two-dimensional real normed space, then each nonempty, closed, and convex subset of E is a nonexpansive retract of E.*

Corollary 4.3. *Each nonempty, closed and convex subset of $\mathbb{R}^2$ endowed with any hyperconvex norm is hyperconvex.*

Remark 4.4. Notice that "any hyperconvex norm on $\mathbb{R}^2$" means essentially (i.e., up to an isometric isomorphism) the maximum norm; this follows from Theorem 2.6 and can also be proved using a geometric argument (see [19, Theorem 4.1]).

Theorem 4.5. *Let E be a hyperconvex normed space. If E is not isometrically isomorphic to $\mathbb{R}^1$ or $(\mathbb{R}^2, \|\cdot\|_\infty)$, then there exists a two-dimensional linear subspace of E which is not hyperconvex.*

Proof. Since E is not isometrically isomorphic to $\mathbb{R}^1$, its dimension must be at least 2. Further, since the only (up to an isometric isomorphism) two-dimensional hyperconvex space is $(\mathbb{R}^2, \|\cdot\|_\infty)$, we may assume $\dim E \geq 3$. By Theorem 2.6 we may assume that E is the space $\mathcal{C}_{\mathbb{R}}(K)$ for some Hausdorff, compact and extremally disconnected topological space K. Since $\dim E \geq 3$, the space K has at least three points, so $\mathcal{C}_{\mathbb{R}}(K)$ includes a copy of $(\mathbb{R}^3, \|\cdot\|_\infty)$. This means that it is enough to prove the theorem in case of $E = \mathbb{R}^3$ with the "maximum" norm.

For simplicity, we will construct an *affine* non-hyperconvex subspace of E; by an appropriate translation one can obtain a *linear* one. Let $V := \{(x_1, x_2, x_3) \in E \mid x_1 + x_2 + x_3 = 1\}$. Consider the following three balls in V: $\overline{B}_V((-1,1,1),1)$, $\overline{B}_V((1,-1,1),1)$, $\overline{B}_V((1,1,-1),1)$. Since the corresponding balls in E intersect only at $(0,0,0) \notin V$, the space V is not hyperconvex. □

Corollary 4.3 and Theorem 4.5 yield the following characterization.

Corollary 4.6. *Let E be a real normed space. The following conditions are equivalent:*

(1) *each nonempty, closed, and convex subset of E is hyperconvex;*

(2) *E is isometrically isomorphic to $\mathbb{R}^1$ or $(\mathbb{R}^2, \|\cdot\|_\infty)$.*

We will now turn our attention to the problem of describing the spaces in which hyperconvexity implies convexity. We will start with an observation suggested to us by Grzybowski [22].

Proposition 4.7. *If a real normed space E is strictly convex, then all its hyperconvex subsets are one-dimensional.*

Proof. Let $A \subset E$ be at least two-dimensional. Therefore there exist three noncollinear points $a, b, c \in A$. Put $p := (1/2)(\|a-b\| + \|b-c\| + \|a-c\|)$ and let $r_a := p - \|b-c\|$, $r_b := p - \|a-c\|$, $r_c := p - \|a-b\|$. It is clear that $\|a-b\| = r_a + r_b$ and similarly for other distances. But E is strictly convex, so we have $\overline{B}_E(a, r_a) \cap \overline{B}_E(b, r_b) = \{(r_b/(r_a+r_b))a + (r_a/(r_a+r_b))b\}$ and $\overline{B}_E(a, r_a) \cap \overline{B}_E(c, r_c) = \{(r_c/(r_a+r_c))a + (r_a/(r_a+r_c))c\}$, so $\overline{B}_E(a, r_a) \cap \overline{B}_E(b, r_b) \cap \overline{B}_E(c, r_c) = \emptyset$. It must be therefore $\overline{B}_A(a, r_a) \cap \overline{B}_A(b, r_b) \cap \overline{B}_A(c, r_c) = \emptyset$, which finishes the proof. □

Corollary 4.8. *If a real normed space E is strictly convex, then all its hyperconvex subsets are convex.*

Proof. From Proposition 4.7 we know that hyperconvex subsets of E are one dimensional; but from Proposition 2.5 we infer that hyperconvex sets are connected, which for one-dimensional sets is equivalent to their convexity. □

To prove the inverse implication, we will need a simple lemma.

Lemma 4.9 (see [23, page 44, Lemma 15.1]). *Let X be a metric space and $a, b, c \in X$ be such that $d(a,c) + d(c,b) = d(a,b)$. If there exist metric segments: S_{ac}, joining the points a and c and S_{cb}, joining the points c and b, then $S_{ac} \cup S_{cb}$ is a metric segment joining the points aand b.*

Now we are ready to prove the following theorem.

Theorem 4.10. *If all hyperconvex subsets of a real normed space E are convex, then E is strictly convex.*

Proof. Assume that E is not strictly convex; we will construct a nonconvex, hyperconvex subset of E. There exist points $a, b, c_1, c_2 \in E$ and positive numbers α, β such that $c_1 \neq c_2$, $\alpha + \beta = 1$ and the equalities $d(a, c_1) = d(a, c_2) = \alpha d(a, b)$ and $d(c_1, b) = d(c_2, b) = \beta d(a, b)$ hold. From Lemma 4.9, both sets $[a, c_1] \cup [c_1, b]$ and $[a, c_2] \cup [c_2, b]$, where $[x, y]$ means an affine segment with endpoints x, y, are metric segments joining a and b (and hence hyperconvex sets). They cannot be, however, both convex, so at least one of them is the desired counterexample. □

Again, combining Corollary 4.8 and Theorem 4.10, we obtain the following characterization of strictly convex normed spaces.

Theorem 4.11. *A normed space is strictly convex if and only if each its hyperconvex subset is convex.*

5. Krein-Milman Type Theorem

In this short section, we will show that a Krein-Milman type theorem holds for $\mathbb{R}$-trees. It turns out that instead of compactness we only need a weaker boundedness condition.

For completeness, let us state the definition of an extremal point in the setting of $\mathbb{R}$-trees.

Definition 5.1. Let X be a subset of an $\mathbb{R}$-tree T. We call a point $x \in X$ an *extremal* point of X if no open metric segment included in X contains x.

Theorem 5.2. *A complete and bounded $\mathbb{R}$-tree is a convex hull of the set of its extremal points.*

Proof. It is enough to show that each point of X lies on a metric segment joining some two extremal points of X. Let $x \in X$. We may assume that x is not extremal; let $x \in (a, b)_d$. The family of all metric segments having x as one of its endpoints satisfies the assumptions of the Kuratowski-Zorn lemma. Let $[x, c]_d \supset [x, a]_d$ and $[x, d]_d \supset [x, b]_d$ be maximal metric segments containing the respective given metric segments. We will first show that c and d are extremal points.

If, say, c were not extremal, we would have $c \in (e, f)_d$ for some $e, f \in X$, $e \neq f$. Let $[c, x]_d \cap [c, e]_d = [c, e']_d$ and $[c, x]_d \cap [c, f]_d = [c, f']_d$. If $e' \neq c \neq f'$, we would have $(c, e']_d \subset (c, x]_d$ and $(c, f']_d \subset (c, x]_d$, so $c \notin [e', f']_d$; but $[c, e']_d \cap [c, f']_d \subset [c, e]_d \cap [c, f]_d = \{c\}$, so $[c, e']_d \cup [c, f']_d = [e', f']_d$—contradiction. This means that $c = e'$ or $c = f'$; assume $c = e'$. Now $[c, x]_d \cap [c, e]_d = \{c\}$, so $[c, x]_d \cup [c, e]_d = [x, e]_d$, which contradicts the maximality of $[x, c]_x$.

Now let us show that $x \in [c, d]_d$. We will prove that $[x, c]_d \cap [x, d]_d = \{x\}$. Assume $[x, c]_d \cap [x, d]_d = [x, y]_d$ and $x \neq y$. Let $\epsilon = \min\{d(x, a), d(x, b)\}$. Choose $w \in (x, y]_d$ such that $d(x, w) < \epsilon$. We have $[x, w]_d \subset [x, c]_d$ and hence $[x, w]_d \subset [x, a]_d$; analogously, $[x, w]_d \subset [x, b]_d$. This means that $w \in [x, a]_d \cap [x, b]_d$ and $w \neq x$; but $[x, a]_d \cap [x, b]_d = \{x\}$—contradiction. □

Since closed and convex subsets of an $\mathbb{R}$-tree are hyperconvex (Proposition 3.3), Corollary 4.6 might give the impression that $\mathbb{R}$-trees are somehow similar to 1- or 2-dimensional vector spaces and that completeness and boundedness of an $\mathbb{R}$-tree imply its compactness. As the following example shows, this analogy is misleading.

Example 5.3. Let T be $\mathbb{R}^2$ with the radial metric. It is easy to see that X is an $\mathbb{R}$-tree and so is $\overline{B}_X((0, 0), 1)$, which is both complete and bounded, but not compact.

6. Hyperconvexity and Measures of Noncompactness

Let us begin this section with the following definition.

Definition 6.1. Let $A \in \mathbb{R}^2$ be some point in the Euclidean plane. Let us define a function $\widetilde{d}_r^A : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,+\infty)$ as follows:

$$\widetilde{d}_r(v_1, v_2) = \begin{cases} \rho(v_1, A) + \rho(v_2, A) & \text{if } v_1 \neq v_2, \\ 0 & \text{if } v_1 = v_2, \end{cases} \tag{6.1}$$

for all $v_1, v_2 \in \mathbb{R}^2$. If $A = (0,0)$, we will write $\widetilde{d}_r$ instead of $\widetilde{d}_r^A$.

It is easy to prove the following lemma.

Lemma 6.2. *$(\mathbb{R}^2, \widetilde{d}_r)$ is a complete metric space.*

We will call the function $\widetilde{d}_r$ (resp., $\widetilde{d}_r^A$) introduced in Definition 6.1, the *modified radial metric* (resp., *centered at* A).

Remark 6.3. The topology of $\mathbb{R}^2$ with the metric $\widetilde{d}_r$ is strictly stronger than the topology of the same space induced by the radial metric.

Lemma 6.4. *The space $\mathbb{R}^2$ with the metric $\widetilde{d}_r$ is not hyperconvex.*

Proof. Let us consider two closed balls $\overline{B}((0,0),1)$ and $\overline{B}((\sqrt{2},\sqrt{2}),1)$. Then

$$\widetilde{d}_r\Big((0,0),\big(\sqrt{2},\sqrt{2}\big)\Big) = 2, \tag{6.2}$$

but

$$\overline{B}((0,0),1) \cap \overline{B}\Big(\big(\sqrt{2},\sqrt{2}\big),1\Big) = \emptyset. \tag{6.3}$$

This shows that the metric $\widetilde{d}_r$ fails to be hyperconvex. $\square$

Now we are going to examine the measures of noncompactness in the space $(\mathbb{R}^2, \widetilde{d}_r)$. For this purpose we are going to use a similar approach as in the case of the measures of noncompactness in $\mathbb{R}^2$ with the radial metric (cf. [20, Theorem 4]). First let us introduce the following definition.

Definition 6.5. Let D be a bounded subset of $(\mathbb{R}^2, \widetilde{d}_r)$. We say that $w' \in \mathbb{R}_+$ satisfies

(1) $V^*(D)$ condition, if for every $w < w'$, there exist infinitely many pairwise distinct points $v \in D$ such that $w < \rho(v,(0,0)) \leq w'$;

(2) $V_*(D)$ condition, if for every $w > w'$, there exist infinitely many pairwise distinct points $v \in D$ such that $w > \rho(v,(0,0)) \geq w'$.

Let us put $v^*(D) = \sup\{0\} \cup \{w' : w' \text{ satisfies } V^*(D) \text{ or } V_*(D)\}$.

Using above conditions we can prove the following theorem.

Theorem 6.6. *For any bounded subset D of $\mathbb{R}^2$ with the metric $\tilde{d}_r$ we have $\alpha(D) = 2v^*(D)$ and $\beta(D) = v^*(D)$.*

Proof. If there exists no nonnegative number w' satisfying either $V^*(D)$ or $V_*(D)$, then clearly D consists of a finite number of points. Hence $\alpha(D) = \beta(D) = 0$ in this case.

Now consider a bounded set D such that there exists a w' satisfying $V^*(D)$ or $V_*(D)$ condition. To prove that $\alpha(D) = 2v^*(D)$, let us first show that $\alpha(D) \geq 2v^*(D)$. For this, consider a covering $(D_j)_{j=1,2,\ldots,m}$ of D such that

$$\max_{j=1,\ldots,m} \delta(D_j) \leq \epsilon \tag{6.4}$$

for some $\epsilon > 0$. Consider the sets $A_n = \{v = (x,y) \in D : \sqrt{x^2+y^2} \geq v^*(D) - 1/n\}$, where $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there exists a $j_n \in \{1,2,\ldots,m\}$ and $v_1^n, v_2^n \in D$ such that $v_1^n \neq v_2^n$, $v_1^n, v_2^n \in D_{j_n} \cap A_n$. Since $\tilde{d}_r(v_1^n, v_2^n) = \rho(v_1^n, 0) + \rho(v_2^n, 0) \geq 2v^*(D) - 2/n$ for every $n \in \mathbb{N}$, $\epsilon \geq 2v^*(D)$. Hence $\alpha(D) \geq 2v^*(D)$.

Next we prove that $\beta(D) \leq v^*(D)$. Obviously, if

$$v^*(D) = \sup_{(x,y)\in D} \rho((x,y),(0,0)), \tag{6.5}$$

then D is contained in the closed ball of center $(0,0)$ and radius $v^*(D)$. So in this case $\beta(D) \leq v^*(D)$.

Let

$$v^*(D) < \sup_{(x,y)\in D} \rho((x,y),(0,0)), \tag{6.6}$$

then according to Definition 6.5, for every $\epsilon > 0$, there exist at most finitely many points $(x,y) \in D$ with the property $\rho((x,y),(0,0)) > v^*(D) + \epsilon$. Hence $\beta(\{(x,y) \in D : \rho((x,y),(0,0)) > v^*(D) + \epsilon\}) = 0$. Moreover,

$$\beta(\{(x,y) \in D : \rho((x,y),(0,0)) \leq v^*(D) + \epsilon\}) \leq v^*(D) + \epsilon. \tag{6.7}$$

Since $\epsilon > 0$ is arbitrary, we get $\beta(D) \leq v^*(D)$ in this case. Finally, we get $v^*(D) \leq (1/2)\alpha(D) \leq \beta(D) \leq v^*(D)$. This implies $\alpha(D) = 2v^*(D)$ and $\beta(D) = v^*(D)$. □

Example 6.7. Using the previous formulae, we can calculate that in $(\mathbb{R}^2, \tilde{d}_r)$ we have $\alpha(\overline{B}((0,0),1)) = 2\beta(\overline{B}((0,0),1)) = 2$; in particular, the closed unit ball is noncompact.

Remark 6.8. It is known (see [12, page 135] for the details) that if a space is hyperconvex, then for any of its bounded subset D, the following equality holds

$$\alpha(D) = 2\beta(D). \tag{6.8}$$

The above theorem shows that even in the nontrivial cases (i.e., in cases, when bounded sets are not necessarily relatively compact), the above equality does not have to imply that the space in question is hyperconvex.

Definition 6.1 can be slightly modified. Namely, let us introduce the following definition.

Definition 6.9. Let us define a function $\overline{d}_r : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,+\infty)$ as follows:

$$\overline{d}_r(v_1,v_2) = \begin{cases} |x_1| + |y_1| + |x_2| + |y_2| & \text{if } v_1 \neq v_2, \\ 0 & \text{if } v_1 = v_2, \end{cases} \tag{6.9}$$

for all $v_1 = (x_1,y_1),\ v_2 = (x_2,y_2) \in \mathbb{R}^2$.

Remark 6.10. It can be easily checked that $(\mathbb{R}^2,\overline{d}_r)$ is a complete metric space. Its topology is also stronger than the topology of $\mathbb{R}^2$ with the radial metric. On the other hand this topology is obviously equivalent to the topology induced by the metric $\widetilde{d}_r$.

Lemma 6.11. *The space $\mathbb{R}^2$ with the metric $\overline{d}_r$ is not hyperconvex.*

Proof. Let us consider two closed balls $\overline{B}((0,0),1)$ and $\overline{B}((2,0),1)$. Then

$$\overline{d}_r((0,0),(2,0)) = 2 \text{ but } \overline{B}((0,0),1) \cap \overline{B}((2,0),1) = \emptyset. \tag{6.10}$$

It shows that the metric $\overline{d}_r$ fails to be hyperconvex. □

For the measures of noncompactness in the space of bounded subsets in the space $(\mathbb{R}^2,\overline{d}_r)$ we have similar formulas to those given in Theorem 6.6.

Definition 6.12. Let D be a bounded subset of $\mathbb{R}^2$ with the metric $\overline{d}_r$. We say that $w' \in \mathbb{R}_+$ satisfies

(1) $U^*(D)$ condition, if for every $w < w'$, there exist infinitely many pairwise distinct points $u = (u_x,u_y) \in D$ such that $w < |u_x| + |u_y| \leq w'$;

(2) $U_*(D)$ condition, if for every $w > w'$, there exist infinitely many pairwise distinct points $u = (u_x,u_y) \in D$ such that $w > |u_x| + |u_y| \geq w'$.

Let us put $u^*(D) = \sup\{0\} \cup \{w' : w' \text{ satisfies } U^*(D) \text{ or } U_*(D)\}$.

Theorem 6.13. *For any bounded subset D of $\mathbb{R}^2$ with the metric $\overline{d}_r$ one has $\alpha(D) = 2u^*(D)$ and $\beta(D) = u^*(D)$.*

The proof of Theorem 6.13 is similar to the proof of Theorem 6.6 and therefore we omit it.

The metric we are going to consider to the end of this section is, roughly speaking, like between the radial metric and the river metric. We will call it a *modified river metric*.

Definition 6.14. Let $A = (a_x, a_y) \in \mathbb{R}^2$. Define a function $d_{\mathrm{ri}}^A : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d_{\mathrm{ri}}^A(v_1, v_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |x_1 - a_x| + |y_1 - a_y| + |x_2 - a_x| + |y_2 - a_y|, & \text{otherwise,} \end{cases} \tag{6.11}$$

for all $v_1 = (x_1, y_1), v_2 = (x_2, y_2) \in \mathbb{R}^2$. If $A = (0, 0)$, we will write d_{ri}^m instead of d_{ri}^A.

The following fact can be easily checked.

Lemma 6.15. *$(\mathbb{R}^2, d_{ri}^m)$ is a complete metric space.*

Remark 6.16. The topology of $(\mathbb{R}^2, d_{\mathrm{ri}}^m)$ is strictly stronger than the topology of $\mathbb{R}^2$ induced by the river metric.

It is interesting to consider a closed ball $\overline{B}((a,b); r) \subset (\mathbb{R}^2, d_{\mathrm{ri}}^m)$, where $a \in \mathbb{R} \setminus \{0\}$ and $|a| + |b| < r < 2|a| + |b|$. Such a ball consists of two disjoint closed sets (a square and a segment) which, in particular, means that it is not connected.

Lemma 6.17. *The space $\mathbb{R}^2$ with the metric d_{ri}^m is not hyperconvex.*

Proof. Let us consider two closed balls $\overline{B}_1((1,1), 3/2)$ and $\overline{B}_2((0,0), 1/2)$. Then $d_{\mathrm{ri}}^m((0,0),(1,1)) = 2$ but $\overline{B}_2((0,0), 1/2) \cap \overline{B}_1((1,1), 3/2) = \emptyset$. This shows that $(\mathbb{R}^2, d_{\mathrm{ri}}^m)$ is not hyperconvex. □

To evaluate the measures of noncompactness of any bounded subset of $(\mathbb{R}^2, d_{\mathrm{ri}}^m)$ one can use a similar approach as in the case of $(\mathbb{R}^2, \overline{d}_r)$ (cf. Definition 6.12 and Theorem 6.13).

In connection with Remark 6.8 let us notice that $(\mathbb{R}^2, \overline{d}_r)$ as well as $(\mathbb{R}^2, d_{\mathrm{ri}}^m)$ are also examples of metric spaces such that $\alpha(D) = 2\beta(D)$ for any bounded subset $D \subset (\mathbb{R}^2, \overline{d}_r)$ or $D \subset (\mathbb{R}^2, d_{\mathrm{ri}}^m)$, but those spaces are not hyperconvex.

7. Generalized Modified Radial and River Metrics

The metric spaces $(\mathbb{R}^2, d_r)$ as well as $(\mathbb{R}^2, d_{\mathrm{ri}})$ are special cases of a general construction provided in [19]. More precisely, let E be a normed space and $C \subset E$ its Chebyshev subset.

Definition 7.1. Let $C \subset E$ be a Chebyshev set in a normed space E and let d_C be any metric defined on C. Let us define $d : E \times E \to [0, +\infty)$ by the formula

$$d(x,y) = \begin{cases} \|x - y\|, & \text{if } P_C(x) = P_C(y), \text{ and } x, P_C(x), y \text{ are collinear,} \\ \|x - P_C(x)\| + d_C(P_C(x), P_C(y)) & \\ \quad + \|P_C(y) - y\|, & \text{otherwise.} \end{cases} \tag{7.1}$$

The above defined function d is a metric (see [19, Lemma 3.1]). Now, the following question can be risen. Is it possible to consider two disjoint Chebyshev sets, instead of one Chebyshev set C, in such a way to get a variant of the metric defined above? The following two examples show that in the case of classical hyperconvex metrics: the radial metric as well as the river metric, this problem seems not to be easy.

Example 7.2. Let $\overline{AB}$ be a fixed segment in $\mathbb{R}^2$ and L the perpendicular bisector of $\overline{AB}$ dividing the whole plane $\mathbb{R}^2$ into two open half-planes $II \ni A$ and $I \ni B$. Let us define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d(v_1, v_2) = \begin{cases} d_r^A(v_1, v_2) & \text{if } v_1, v_2 \in II \cup L, \\ d_r^B(v_1, v_2) & \text{if } v_1, v_2 \in I \cup L, \\ \rho(v_1, A) + \rho(A, B) + \rho(B, v_2) & \text{if } v_1 \in II, v_2 \in I, \\ \rho(v_2, A) + \rho(A, B) + \rho(B, v_1) & \text{if } v_1 \in I, v_2 \in II, \end{cases} \tag{7.2}$$

for all $v_1,\ v_2 \in \mathbb{R}^2$, where d_r^A, d_r^B are the radial metrics on the plane centered at A and B, respectively. Then this d is not a metric. Indeed it does not satisfy the triangle inequality in the following case.

Let us consider three points $v_1, v_2, v_3 \in \mathbb{R}^2$ such that $v_1 \in II,\ v_2 \in L,\ v_3 \in I$; $v_2,\ v_1$, and A are collinear; $v_2,\ v_3$, and B are collinear; $\rho(v_2, v_1) < \rho(v_1, A)$ and $\rho(v_2, v_3) < \rho(v_3, B)$. Then $d(v_1, v_2) + d(v_2, v_3) < d(v_1, v_3)$.

Example 7.3. Let $A := (-a, 0)$ and $B := (a, 0)$, where $a > 0$, be two points in $\mathbb{R}^2$. Let L be the perpendicular bisector of $\overline{AB}$; it divides the whole plane $\mathbb{R}^2$ into two open half-planes $II \ni A$ and $I \ni B$. Let us define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d(v_1, v_2) = \begin{cases} d_{\mathrm{ri}}(v_1, v_2) & \text{if } v_1, v_2 \in I \cup L \\ & \quad \text{or } v_1, v_2 \in II \cup L, \\ d_{\mathrm{ri}}(v_1, A) + \rho(A, B) + d_{\mathrm{ri}}(B, v_2) & \text{if } v_1 \in II, v_2 \in I, \\ d_{\mathrm{ri}}(v_2, A) + \rho(A, B) + d_{\mathrm{ri}}(B, v_1) & \text{if } v_1 \in I, v_2 \in II, \end{cases} \tag{7.3}$$

for all $v_1,\ v_2 \in \mathbb{R}^2$, where d_{ri} denotes the river metric. Then this d is not a metric. Indeed, it does not satisfy the triangle inequality in the following case. Let $A = (-2, 0),\ B = (2, 0)$, and let us take three points $v_1 = (-1, 1),\ v_2 = (0, 0),\ v_3 = (1, 1)$. Then, by the definition $d(v_1, v_2) = 2$ and $d(v_2, v_3) = 2$ but $d(v_1, v_3) = 8$, which shows $d(v_1, v_2) + d(v_2, v_3) < d(v_1, v_3)$.

However, it appears that all the metrics introduced in Section 6 (Definitions 6.1, 6.9 and 6.14) are appropriate to define new metrics using the idea described at the beginning of this section.

Let us begin with the following definition.

Definition 7.4. Let $\overline{AB}$ be a segment in $\mathbb{R}^2$ and L be the perpendicular bisector of $\overline{AB}$ dividing the whole plane $\mathbb{R}^2$ into two open half-planes $II \ni A$ and $I \ni B$. Let us define a function $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d_1(v_1, v_2) = \begin{cases} \tilde{d}_r^A(v_1, v_2), & \text{if } v_1, v_2 \in II \cup L, \\ \tilde{d}_r^B(v_1, v_2), & \text{if } v_1, v_2 \in I \cup L, \\ \rho(v_1, A) + \rho(A, B) + \rho(B, v_2), & \text{if } v_1 \in II, v_2 \in I, \\ \rho(v_2, A) + \rho(A, B) + \rho(B, v_1), & \text{if } v_1 \in I, v_2 \in II, \end{cases} \tag{7.4}$$

for all $v_1, v_2 \in \mathbb{R}^2$, where $\tilde{d}_r^A$ and $\tilde{d}_r^B$ denote modified radial metrics centered at A and B, respectively.

Let us note that if v_1 and v_2 both are in L, then $d_1(v_1, v_2) = \tilde{d}_r^A(v_1, v_2) = \tilde{d}_r^B(v_1, v_2)$, so d_1 is well-defined.

Lemma 7.5. $(\mathbb{R}^2, d_1)$ *is a complete metric space.*

Proof. It is easy to check that d_1 is a metric. Now to verify that it is complete, let us consider a Cauchy sequence v_n in the space $(\mathbb{R}^2, d_1)$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, the points v_n belong to the same closed half-plane $I \cup L$ or $II \cup L$. Hence, by Lemma 6.2, (v_n) is convergent, which completes the proof. □

Remark 7.6. It is clear that the topologies of $\mathbb{R}^2$ induced by the metric d_1 and the modified radial metric are not comparable.

Lemma 7.7. *The space* $\mathbb{R}^2$ *with the metric* d_1 *is not hyperconvex.*

Proof. For convenience consider $A = (1, 0)$, $B = (-1, 0)$ and consider two closed balls $\overline{B}((1, 0), 1/2)$ and $\overline{B}((1, 1), 1/2)$. Then $d_1((1, 0), (1, 1)) = 1$ but $\overline{B}((1, 0), 1/2) \cap \overline{B}((1, 1), 1/2) = \emptyset$. □

Remark 7.8. It is easy to evaluate the Kuratowski and Hausdorff measures of noncompactness of bounded sets in $\mathbb{R}^2$ with the metric d_1.

Indeed, let us consider a bounded set D in $\mathbb{R}^2$ with this metric. Then we can write D as the union of two sets U and V, where

$$U = D \cap (I \cup L), \qquad V = D \cap (II). \tag{7.5}$$

Then, by the maximum property of the measures of noncompactness, we get

$$\alpha(D) = \alpha(U \cup V) = \max(\alpha(U), \alpha(V)). \tag{7.6}$$

To evaluate $\alpha(U)$ and $\alpha(V)$ it is enough to apply formulas similar to the one given in Theorem 6.6.

Remark 7.9. It is clear that in Definition 7.4 one can replace $\tilde{d}_r^A$, $\tilde{d}_r^B$ by $\overline{d}_r^A$, $\overline{d}_r^B$, respectively, (cf. Definition 6.9) getting again a complete metric space which is not hyperconvex.

Now, using the metric from Definition 6.14, let us introduce the following metric.

Definition 7.10. Let $\overline{AB}$ be a fixed segment in $\mathbb{R}^2$ parallel to the x-axis and L perpendicular bisector of $\overline{AB}$ dividing the whole plane into two open half-planes I and II. Let us define a function $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d_2(v_1, v_2) = \begin{cases} d_{\mathrm{ri}}^A(v_1, v_2) & \text{if } v_1, v_2 \in II \cup L, \\ d_{\mathrm{ri}}^B(v_1, v_2) & \text{if } v_1, v_2 \in I \cup L, \\ d_{\mathrm{ri}}^A(v_1, A) + \rho(A, B) + d_{\mathrm{ri}}^B(B, v_2) & \text{if } v_1 \in II,\ v_2 \in I, \\ d_{\mathrm{ri}}^B(v_1, B) + \rho(A, B) + d_{\mathrm{ri}}^A(A, v_2) & \text{if } v_1 \in I,\ v_2 \in II, \end{cases} \tag{7.7}$$

for all $v_1,\ v_2 \in \mathbb{R}^2$, where d_{ri}^A and d_{ri}^B denote the metrics from Definition 6.14.

One can prove the following lemma.

Lemma 7.11. *$(\mathbb{R}^2, d_2)$ is a complete metric space.*

The proof of this Lemma is similar to the proof of Lemma 7.5 and therefore we omit it.

Remark 7.12. The metric d_2 is a variant of the metric d_{ri}^m defined in Definition 6.14. The topologies induced by these metrics are not comparable. The space $(\mathbb{R}^2, d_2)$ is not hyperconvex, either. Finally, to find the Kuratowski and the Hausdorff measures of noncompactness of bounded sets in $\mathbb{R}^2$ with the metric d_2, it is enough to use the same approach as in Remark 7.8.

In Definitions 7.4 and 7.10, we considered two Chebyshev sets. Now one can think of the following question. Is it possible to increase the number of suitably chosen Chebyshev sets? The answer is "yes." Let us introduce the following definition.

Definition 7.13. Let us consider the square $ABCD$ in $\mathbb{R}^2$ with vertices: $A := (a, a), B := (-a, a), C := (-a, -a), D := (a, -a)$, where $a > 0$. Denote $L_1 := \{(x, y) \in \mathbb{R}^2 \mid y = 0\}, L_2 := \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$, $L_1^+ := \{(x, y) \in L_1 \mid x \geq 0\}$, $L_1^- := \{(x, y) \in L_1 \mid x \leq 0\}$, $L_2^+ := \{(x, y) \in L_2 \mid y \geq 0\}$, $L_2^- := \{(x, y) \in L_2 \mid y \leq 0\}$. Let d_m be the "maximum" metric on $\mathbb{R}^2$. By d_1^{AB}, d_1^{AC}, and so forth, we will mean a metric defined as in Definition 7.4, but using $d_m(A, B)$, $d_m(A, C)$, and so forth, instead of $\rho(A, B)$, $\rho(A, C)$, and so forth. Denote the four open quadrants by $I \ni A$, $II \ni B$, $III \ni C$ and $IV \ni D$. Let us define a function $d_4 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ as follows:

$$d_3(v_1, v_2) = \begin{cases} \tilde{d}_r^A(v_1, v_2) & \text{if } v_1, v_2 \in I^c, \\ \tilde{d}_r^B(v_1, v_2) & \text{if } v_1, v_2 \in II^c, \\ \tilde{d}_r^C(v_1, v_2) & \text{if } v_1, v_2 \in III^c, \\ d_1^D(v_1, v_2) & \text{if } v_1, v_2 \in IV^c, \\ d_1^{AB}(v_1, v_2) & \text{if } v_1 \in I, v_2 \in II \text{ or vice versa}, \\ d_1^{BC}(v_1, v_2) & \text{if } v_1 \in II, v_2 \in III \text{ or vice versa}, \\ d_1^{CD}(v_1, v_2) & \text{if } v_1 \in III, v_2 \in IV \text{ or vice versa}, \\ d_1^{DA}(v_1, v_2) & \text{if } v_1 \in IV, v_2 \in I \text{ or vice versa}, \\ d_1^{AC}(v_1, v_2) & \text{if } v_1 \in I, v_2 \in III \text{ or vice versa}, \\ d_1^{BD}(v_1, v_2) & \text{if } v_1 \in II, v_2 \in IV \text{ or vice versa}, \\ \tilde{d}_r^A(v_1, A) + d_m(A, B) + \tilde{d}_r^B(B, v_2), & \\ \text{or } \tilde{d}_r^A(v_1, A) + d_m(A, C) + \tilde{d}_r^C(C, v_2), & \\ & \text{if } v_1 \in I, v_2 \in {L_1}^-, \\ \tilde{d}_r^D(v_1, D) + d_m(D, B) + \tilde{d}_r^B(B, v_2), & \\ \text{or } \tilde{d}_r^D(v_1, D) + d_m(D, C) + \tilde{d}_r^C(C, v_2), & \\ & \text{if } v_1 \in IV, v_2 \in {L_1}^-, \end{cases} \tag{7.8}$$

and eight more similar expressions involving L_1^+, L_2^+, and L_2^- for all $v_1, v_2 \in \mathbb{R}^2$, where I^c, II^c, III^c and IV^c denote the closed quadrants and $\tilde{d}_r^A$, $\tilde{d}_r^B$, $\tilde{d}_r^c$, and $\tilde{d}_r^D$ denote the modified radial metrics defined in Definition 6.1.

Lemma 7.14. $(\mathbb{R}^2, d_3)$ *is a complete metric space.*

Proof. To prove that d_3 is a metric on $\mathbb{R}^2$ is straightforward, although quite long, so we omit this proof. To prove that $(\mathbb{R}^2, d_3)$ is complete, let us consider a Cauchy sequence (v_n) in the space $(\mathbb{R}^2, d_3)$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_3(v_m, v_n) < \epsilon$ for every $m, n \geq N$. It means there exists $N \in \mathbb{N}$ such that for every $n \geq N, v_n$ belongs to the same closed quadrant, because if v_n and v_m were in different quadrants (without loss of generality suppose $v_m \in I$ and $v_n \in II$), then

$$d_3(v_m, v_n) = \tilde{d}_r^A(v_m, A) + d_m(A, B) + \tilde{d}_r^B(B, v_n). \tag{7.9}$$

So, if we choose $\epsilon < d_m(A, B)$, then $d_3(v_n, v_m) > \epsilon$ which contradicts that (v_n) is a Cauchy sequence. Hence almost all the terms of any Cauchy sequence must be in the same closed quadrant. Thus by Lemma 6.2, (v_n) is convergent, which shows that the space $(\mathbb{R}^2, d_3)$ is complete. □

For the convenience of the reader, let us present a figure of a closed ball $\overline{B}_B(P,r)$ in $(\mathbb{R}^2, d_3)$, where $A = (1,1)$, $B = (-1,1)$, $C = (-1,-1)$, $D = (1,-1)$, $P = (a,b)$, $a > 1$, $b > 1$, and $\rho(A,P) + d_m(A,B) < r < \rho(A,P) + (3/2)d_m(A,B)$.

Obviously, the following lemma holds.

Lemma 7.15. *The space $\mathbb{R}^2$ with the metric d_3 is not hyperconvex.*

Proof. For convenience let us consider $A = (1,1)$, $B = (-1,1)$, $C = (-1,-1)$, and $D = (1,-1)$ and two closed balls $\overline{B}((1,1),1/2)$ and $\overline{B}((2,1),1/2)$. Then $d_3((1,1),(2,1)) = 1$ but $\overline{B}((1,1),1/2) \cap \overline{B}((2,1),1/2) = \emptyset$. □

Remark 7.16. It is easy to evaluate the Kuratowski and Hausdorff measures of noncompactness of bounded sets in $(\mathbb{R}^2, d_3)$. Indeed, one can use a similar approach as in Remark 7.8.

8. Linking Construction

In this section we will give a slight generalization of the so-called *linking construction* described by Aksoy and Maurizi in [14] and show how this generalization includes the metrics of Section 7. Notice that a similar concept appears in [24], where it is used to study existence of certain mappings between Banach spaces.

Definition 8.1 (cf. [14, page 221, Theorem 2.1]). Let (X,d) be a metric space and $\{W_\lambda, d_\lambda\}_{\lambda\in\Lambda}$ a collection of pairwise disjoint metric spaces, each disjoint with X. Let $f : \Lambda \to X$ be an arbitrary function and let $g : \Lambda \to \bigcup_{\lambda\in\Lambda} W_\lambda$ be a function satisfying $g(\lambda) \in W_\lambda$ for each $\lambda \in \Lambda$. Define $\widetilde{W}_\lambda := W_\lambda \setminus \{g(\lambda)\}$ for $\lambda \in \Lambda$. Let $Z := X \cup \bigcup_{\lambda\in\Lambda} \widetilde{W}_\lambda$. Define the function $d_Z : Z \times Z \to [0,+\infty)$ by the formula

$$d_Z(x,y) := \begin{cases} d(x,y) & \text{if } x,y \in \mathrm{X}, \\ d_\lambda(x,y) & \text{if } x,y \in \mathrm{W}_\lambda \text{ for some } \lambda \in \Lambda, \\ d_{\lambda_1}(x,g(\lambda_1)) + d(f(\lambda_1),f(\lambda_2)) + d_{\lambda_2}(g(\lambda_2),y) & \\ \qquad \text{if } x \in \widetilde{W}_{\lambda_1},\ y \in \widetilde{W}_{\lambda_2}, \lambda_1 \neq \lambda_2 & \\ d(x,f(\lambda)) + d_\lambda(g(\lambda),y) & \\ \qquad \text{if } x \in \mathrm{X}, y \in \widetilde{W}_\lambda \text{ for some } \lambda \in \Lambda, & \\ d(y,f(\lambda)) + d_\lambda(g(\lambda),x) & \\ \qquad \text{if } y \in \mathrm{X} \text{ and } x \in \widetilde{W}_\lambda \text{ for some } \lambda \in \Lambda. & \end{cases} \tag{8.1}$$

Theorem 8.2 (cf. [14, page 221, Theorem 2.1]). *The function d_Z defined above is a metric on Z. If all the metric spaces X, W_λ for $\lambda \in \Lambda$ are hyperconvex, then so is (Z, d_Z).*

Remark 8.3. The paper [14] contains the above theorem only for hyperconvex spaces. It is obvious that d_Z is a metric also in the general case.

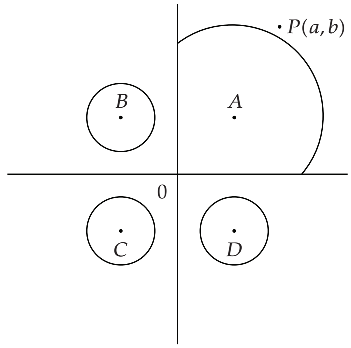

Figure 1: An example of a ball in the metric d_3.

Remark 8.4. The authors of the paper [14] applied their version of Theorem 8.2 to obtain the hyperconvexity of the metric of Definition 7.1 (see [14, Theorem 2.2]). Let us notice that an identical result was given in an earlier work [19].

Proposition 8.5. *The metric d_Z from Definition 8.1 is complete if all the spaces W_λ and X are complete.*

Proof. Let (x_n) be a Cauchy sequence in (Z, d_Z). We will show that (x_n) has a convergent subsequence. If (x_n) has infinitely many terms in Z, we are done. If (x_n) has infinitely many terms in some $\widetilde{W}_\lambda$, it must be convergent in W_λ to some $x \in W_\lambda$; if $x \neq g(\lambda)$, the proof is complete, and if $x = g(\lambda)$, it is easily seen that $x_n \to f(\lambda)$ in Z as $n \to \infty$. Therefore we may assume that (x_n) includes only a finite number (possibly zero) of points from Z and each $\widetilde{W}_\lambda$. Define $P_X : Z \to X$ by

$$P_X(x) := \begin{cases} x & \text{if } x \in X; \\ f(\lambda) & \text{if } x \in \widetilde{W}_\lambda \text{ for some } \lambda \in \Lambda. \end{cases} \tag{8.2}$$

Observe that $\lim_{n\to\infty} d_Z(x_n, P_X(x_n)) = 0$; for if that were not the case, there would exist a subsequence (x_{n_k}) and an $\epsilon > 0$ such that each x_{n_k} would lie in different $\widetilde{W}_\lambda$ and $d_Z(x_{n_k}, P_X(x_{n_k})) > \epsilon$; this would mean that $d_Z(x_{n_k}, x_{n_l})) > 2\epsilon$ for all $k, l \in \mathbb{N}$—contradiction with (x_n) being Cauchy.

Now notice that $d_Z(P_X(x_m), P_X(x_n)) \leq d_Z(x_m, x_n)$ for $m, n \in \mathbb{N}$, so the sequence $(P_X(x_n))$ is also Cauchy and hence convergent to some $x \in X$. We have $d_Z(x, x_n) \leq d_Z(x, P_X(x_n)) + d_Z(P_X(x_n), x_n) \to 0$ as $n \to \infty$ and the proof is complete. □

Remark 8.6. To evaluate the Kuratowski and Hausdorff measures of noncompactness of bounded sets in Z with the metric d_Z, when the set Λ is finite, we use following procedure.

Let us consider a bounded set D in Z with the metric d_Z. Then we can write D as the following union:

$$D = (X \cap D) \cup \left(\bigcup_{\lambda\in\Lambda} \left(\widetilde{W}_\lambda \cap D \right) \right). \tag{8.3}$$

Then, by the maximum property of the measures of noncompactness, we get

$$\begin{aligned}\alpha(D) &= \alpha\left((X\cap D)\cup\left(\bigcup_{\lambda\in\Lambda}\left(\widetilde{W}_\lambda\cap D\right)\right)\right)\\ &= \max_{\lambda\in\Lambda}\left\{\alpha(X\cap D),\alpha\left(\widetilde{W}_\lambda\cap D\right)\right\}.\end{aligned}\tag{8.4}$$

Example 8.7. Notice that the metric from Definition 7.4 can be obtained as a special case of Definition 8.1. Indeed, put $X := \{A, B\}$ and $\Lambda := \mathbb{R}^2 \setminus X$. For each $\lambda \in \Lambda$, define

$$f(\lambda) := \begin{cases} A & \text{if } \lambda \in II, \\ B & \text{if } \lambda \in I \cup L, \end{cases}\tag{8.5}$$

$W_\lambda := \{f(\lambda), \lambda\} \times \{0\}$ for $\lambda \in \Lambda$ and $g(\lambda) := (f(\lambda), 0)$ for $\lambda \in \Lambda$.

In a similar way, other metrics from Sections 6 and 7 are special cases of Definition 8.1. As an example, let us provide a way to construct the metric d^m_{ri} from Definition 6.14.

Example 8.8. Let $X := \mathbb{R}$ and $W_\lambda := \{\lambda\} \times \mathbb{R}$ for $\lambda \in \Lambda := \mathbb{R}$. Define the metric $d : X \times X \to [0, +\infty)$ by the formula

$$d(x,y) := \begin{cases} |x| + |y| & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}\tag{8.6}$$

For each $\lambda \in \Lambda$, let $d_\lambda : W_\lambda \times W_\lambda \to [0, +\infty)$ be the metric defined by $d_\lambda((\lambda, x), (\lambda, y)) := |x-y|$. Further, let $f : \Lambda \to X$ be an identity mapping and $g : \Lambda \to \bigcup_{\lambda\in\Lambda} W_\lambda : \lambda \mapsto (\lambda, 0)$. It is easily seen that applying Definition 8.1 we obtain the metric space d^m_{ri}.

At the beginning of Section 7 we posed a question whether it is possible to construct a metric analogous to that from Definition 7.1, but with more than one Chebyshev subset. In all our examples, however, these subsets were singletons. Let us now show an example of two similar metrics constructed using two disjoint Chebyshev subsets consisting of more than one point.

Example 8.9. Define the following two Chebyshev subsets of the Euclidean plane: $C_- := \operatorname{conv}\{(-1,-1),(-1,1)\}$ and $C_+ := \operatorname{conv}\{(1,-1),(1,1)\}$. Put $\Lambda := X := C_- \cup C_+$. Let $H_- := \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ and $H_+ := \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$. Let $P_- : H_- \to C_-$ and $P_+ : H_+ \to C_+$ be metric projections and define $P : \mathbb{R}^2 \to X$ by the formula

$$P(x) := \begin{cases} P_-(x) & \text{if } x \in H_-, \\ P_+(x) & \text{if } x \in H_+. \end{cases}\tag{8.7}$$

For each $\lambda \in \Lambda$, let $W_\lambda := \{x \in \mathbb{R}^2 \mid P(x) = \lambda\} \times \{0\}$. Let $f : \Lambda \to X$ the identity map and $g : \Lambda \to \mathbb{R}^2 \times \{0\}$ be defined by $g(\lambda) := (\lambda, 0)$ for $\lambda \in \Lambda$. The metrics on X and W_λ's are inherited from $\mathbb{R}^2$. Applying Definition 8.1 we obtain a certain metric on $\mathbb{R}^2$. Let us notice that it is not complete; taking $\Lambda := \mathbb{R}^2$ and $W_\lambda := \{\lambda, P(\lambda)\} \times \{0\}$ for $\lambda \in \Lambda$, $f := P$ and g as before we obtain another metric, this time complete. Let us finish by observing that since X, and hence Z, is disconnected, in both cases Z cannot be hyperconvex.

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