Research Article

Weak ψ-Sharp Minima in Vector Optimization Problems

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We present a sufficient and necessary condition for weak ψ-sharp minima in infinite-dimensional spaces. Moreover, we develop the characterization of weak ψ-sharp minima by virtue of a nonlinear scalarization function.

1. Introduction

The notion of a weak sharp minimum in general mathematical program problems was first introduced by Ferris in [1]. It is an extension of sharp minimum in [2]. Weak sharp minima play important roles in the sensitivity analysis [3, 4] and convergence analysis of a wide range of optimization algorithms [5]. Recently, the study of weak sharp solution set covers real-valued optimization problems [5–8] and piecewise linear multiobjective optimization problems [9–11].

Most recently, Bednarczuk [12] defined weak sharp minima of order $m$ for vector-valued mappings under an assumption that the order cone is closed, convex, and pointed and used the concept to prove upper Hölderess and Hölder calmness of the solution set-valued mappings for a parametric vector optimization problem. In [13], Bednarczuk discussed the weak sharp solution set to vector optimization problems and presented some properties in terms of well-posedness of vector optimization problems. In [14], Studniarski gave the definition of weak $ψ$-sharp local Pareto minimum in vector optimization problems under the assumption that the order cone is convex and presented necessary and sufficient conditions under a variety of conditions. Though the notions in [12, 14] are different for vector optimization problems, they are equivalent for scalar optimization problems. They are a generalization of the weak sharp local minimum of order $m$.

In this paper, motivated by the work in [14, 15], we present a sufficient and necessary condition of which a point is a weak $ψ$-sharp minimum for a vector-valued mapping in the
infinite-dimensional spaces. In addition, we develop the characterization of weak \( q \)-sharp minima in terms of a nonlinear scalarization function.

This paper is organized as follows. In Section 2, we recall the definitions of the local Pareto minimizer and weak \( q \)-sharp local minimizer for vector-valued optimization problems. In Section 3, we present a sufficient and necessary condition for weak \( q \)-sharp local minimizer of vector-valued optimization problems. We also give an example to illustrate the optimality condition.

### 2. Preliminary Results

Throughout the paper, \( X \) and \( Y \) are normed spaces. \( B(x, \delta) \) denotes the open ball with center \( x \in X \) and radius \( \delta > 0 \). \( \mathcal{N}(x) \) is the family of all neighborhoods of \( x \), and \( \text{dist}(x, W) \) is the distance from a point \( x \) to a set \( W \subset X \). The symbols \( S^c \), \( \text{int} S \) and \( \text{bds} S \) denote, respectively, the complement, interior and boundary of \( S \).

Let \( D \subset Y \) be a convex cone (containing \( 0 \)). The cone defines an order structure on \( Y \), that is, a relation \( \leq \) in \( Y \times Y \) is defined by \( y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in D \). \( D \) is a proper cone if \( \{0\} \neq D \neq Y \).

Let \( \Omega \) be an open subset of \( X \), \( S \subset \Omega \). Given a vector-valued map \( f : \Omega \to Y \), the following abstract optimization is considered:

\[
\text{Min}\{f(x) : x \in S\}. \tag{2.1}
\]

In the sequel, we always assume that \( D \) is a proper closed and convex cone.

**Definition 2.1.** One says that \( x_0 \) is a local Pareto minimizer for (2.1), denoted by \( x_0 \in \text{LMin}(f, S) \), if there exists \( U \in \mathcal{N}(x) \) for which there is no \( x \in S \cap U \) such that

\[
f(x) - f(x_0) \in (-D) \setminus D. \tag{2.2}
\]

If one can choose \( U = X \), one will say that \( x_0 \) is a Pareto minimizer for (2.1), denoted by \( x_0 \in \text{Min}(f, S) \).

Note that (2.2) may be replaced by the simple condition \( f(x) - f(x_0) \in (-D) \setminus \{0\} \) if we assume that the cone \( D \) is pointed.

**Definition 2.2** (see [14]). Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be a nondecreasing function with the property \( \varphi(t) = 0 \Leftrightarrow t = 0 \) (such a family of functions is denoted by \( \Psi \)). Let \( x_0 \in S \). One says that \( x_0 \) is a weak \( \varphi \)-sharp local Pareto minimizer for (2.1), denoted by \( x_0 \in \text{WSL}(\varphi, f, S) \), if there exist a constant \( \alpha > 0 \) and \( U \in \mathcal{N}(x_0) \) such that

\[
(f(x) + D) \cap B(f(x_0), \alpha \varphi(\text{dist}(x, W))) = \emptyset, \quad \forall x \in (S \cap U) \setminus W, \tag{2.3}
\]

where

\[
W := \{x \in S : f(x) = f(x_0)\}. \tag{2.4}
\]
If one can choose \( U = X \), one says \( x_0 \) is a weak \( q \)-sharp minimizer for (2.1), denoted by \( x_0 \in \text{WS}(q,f,S) \). In particular, let \( q_m(t) := t^m \) for \( m = 1, 2, \ldots \). Then, one says that \( x_0 \) is a weak \( q \)-sharp local Pareto minimizer of order \( m \) for (2.1) if \( x_0 \in \text{WSL}(q_m,f,S) \), and one says that \( x_0 \) is a weak sharp Pareto minimizer of order \( m \) for (2.1) if \( x_0 \in \text{WS}(q_m,f,S) \).

Remark 2.3. If \( W \) is a closed set, condition (2.3) can be expressed as the following equivalent forms:

\[
\begin{align*}
 f(x) & \in \left( f(x_0) + B(0, \alpha \psi(\text{dist}(x,W))) - D \right)^c, \quad \forall x \in (S \cap U) \setminus W, \quad (2.5) \\
 d(f(x) - f(x_0), -D) & \geq \alpha \psi(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W. \quad (2.6)
\end{align*}
\]

Remark 2.4. In the Definition 2.2, if \( Y = R, D = [0, +\infty) \), and \( q = q_m \), then the relation (2.6) becomes the following form:

\[
 f(x) - f(x_0) \geq \alpha(\text{dist}(x,W))^m, \quad \forall x \in S \cap U, \quad (2.7)
\]

which is the well-known definition of a weak sharp minimizer of order \( m \) for (2.1); see [16].

3. Main Results

In this section, we first generalize the result of Theorem 1 in Studniarski [14] to infinite-dimensional spaces. Finally, we develop the characterization of weak \( q \)-sharp minimizer by means of a nonlinear scalarization function.

Let \( D \subset Y \) be a proper closed convex cone with \( \text{int} D \neq \emptyset \). The topological dual space of \( Y \) is denoted by \( Y^* \). The polar cone to \( D \) is \( D^* = \{ \lambda \in Y^* : \langle \lambda, y \rangle \geq 0, \forall y \in D \} \). It is well known that the cone \( D^* \) contains a \( \omega^* \)-compact convex set \( \Lambda \) with \( 0 \notin \Lambda \) such that

\[
D^* = \text{cone} \Lambda = \{ r\lambda : r \geq 0, \lambda \in \Lambda \}. \quad (3.1)
\]

The set \( \Lambda \) is called a base for the dual cone \( D^* \). Recall that a point \( \lambda \) is an extremal point of a set \( \Lambda \) if there exist no different points \( \lambda_1, \lambda_2 \in \Lambda \) and \( t \in (0,1) \) such that \( \lambda = t\lambda_1 + (1-t)\lambda_2 \).

Theorem 3.1. Suppose that \( f : X \to Y \) is a vector-valued map. Let \( D \subset Y \) be a proper closed convex cone with \( \text{int} D \neq \emptyset \), \( x_0 \in S \), and \( \varphi \in \Psi \).

(i) Let \( \Lambda \) be a \( \omega^* \)-compact convex base of \( D^* \) and \( Q \) the set of extremal points of \( \Lambda \). Suppose that \( W \) defined by (2.4) is a closed set. Then, \( x_0 \in \text{WSL}(q,f,S) \) if and only if there exist \( U \in \mathcal{M}(x) \), a constant \( \alpha > 0 \), a covering \( \{ S_\lambda : \lambda \in Q \} \) of \( S \cap U \), and

\[
\langle \lambda, f(x) \rangle > \langle \lambda, f(x_0) \rangle + \alpha \psi(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W, \forall \lambda \in Q. \quad (3.2)
\]

(ii) Let \( Q \subset D^* \setminus \{ 0 \} \) and assume that \( D^* = \text{cl cone} \text{co} Q \). Then \( x_0 \in \text{LMin}(f,S) \) if and only if there exists a covering \( \{ S_\lambda : \lambda \in Q \} \) of \( S \cap U \) such that

\[
\langle \lambda, f(x) \rangle > \langle \lambda, f(x_0) \rangle, \quad \forall x \in (S \cap U) \setminus W, \forall \lambda \in Q. \quad (3.3)
\]
Proof. (i) Part “only if”: by assumption, there exist \( \beta > 0 \) and \( U \in \mathcal{N}(x_0) \) such that
\[
(f(x) - f(x_0) + D) \cap B(0, \beta \varphi(\text{dist}(x, W))) = \emptyset, \quad \forall x \in (S \cap U) \setminus W. \tag{3.4}
\]

Let \( e \in \text{int } D \) be a fixed point. Set \( \beta_0 = \inf_{\lambda \in \Lambda} \langle \lambda, e \rangle \). Since \( \Lambda \) is \( w^* \)-compact, the infimum is attained at a point of \( Q \). Namely, \( \beta_0 = \min_{\lambda \in Q} \langle \lambda, e \rangle \). Clearly, \( \langle \lambda, e \rangle > 0 \) for any \( \lambda \in \Lambda \). Hence, \( \beta_0 > 0 \).

For each \( \lambda \in Q \), we define
\[
S_\lambda = \left\{ x \in S \cap U : \langle \lambda, f(x) \rangle \geq \langle \lambda, f(x_0) \rangle + \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))\beta_0 \right\}. \tag{3.5}
\]

We will show that
\[
S \cap U \subset \bigcup_{\lambda \in Q} S_\lambda. \tag{3.6}
\]

Let \( x \in S \cap U \). If \( x \in W \), then \( f(x) = f(x_0) \) by (2.4), hence, \( x \in S_\lambda \) for all \( \lambda \in Q \). If \( x \not\in W \), suppose that \( x \not\in S_\lambda \) for any \( \lambda \in Q \), then
\[
\langle \lambda, f(x) \rangle < \langle \lambda, f(x_0) \rangle + \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))\beta_0, \quad \forall \lambda \in Q. \tag{3.7}
\]

This relation, together with statement \( \langle \lambda, e \rangle \geq \beta_0 \) yields
\[
\left\langle \lambda, f(x_0) - f(x) + \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))e \right\rangle > 0, \quad \forall \lambda \in Q. \tag{3.8}
\]

Obviously, for any \( \lambda \in D^* \), the above relation becomes the following form:
\[
\left\langle \lambda, f(x_0) - f(x) + \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))e \right\rangle \geq 0. \tag{3.9}
\]

Consequently, by the bipolar theorem, one has
\[
d := f(x_0) - f(x) + \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))e \in D. \tag{3.10}
\]

Therefore,
\[
f(x) - f(x_0) + d = \frac{\beta}{2\|e\|} \varphi(\text{dist}(x, W))e, \tag{3.11}
\]
and \( f(x) - f(x_0) + d \in B(0, \beta \varphi(\text{dist}(x, W))) \), which is a contradiction to (3.4). We have thus proved that \( S_\lambda \) covers \( S \cap U \).
Now, let $x \in (S \cap U) \setminus W$ and $\lambda \in Q$. From the procedure of the above proof, we see that $(S \cap U) \setminus W \subset \bigcup_{\lambda \in Q} S_1$. Hence, by (3.5), set $\alpha = \beta \beta_0/(4||e||)$, inequality (3.2) is true.

Part "if": we define $\beta_1 = \sup_{\lambda \in A}(\lambda, e)$. The supremum is attained at an extremal point because of the $w^*$-compactness of $\Lambda$. So $\beta_1 = \max_{\lambda \in Q}(\lambda, e) > 0$ and $\beta_1^{-1}(\lambda, e) \leq 1$ for any $\lambda \in Q$. Hence, by assumption, we have

$$\langle \lambda, f(x) \rangle > \langle \lambda, f(x_0) \rangle + \alpha \psi(\text{dist}(x, W)) \geq \langle \lambda, f(x_0) \rangle + \beta_1^{-1}\alpha \psi(\text{dist}(x, W))(\lambda, e),$$

for $x \in (S \cap U) \setminus W$ and $\lambda \in Q$.

Now, suppose that for all $\beta > 0$, (3.4) is false, then there exist $x' \in (S \cap U) \setminus W$ and $d \in D$ such that

$$f(x') - f(x_0) + d \in B(0, \beta \psi(\text{dist}(x, W))).$$

Let $e \in \text{int} D$ be a fixed point, and since $D$ is a cone, there is $k > 0$ such that $B(0, 1) \subset ke - D$. Consequently,

$$B(0, \beta \psi(\text{dist}(x, W))) \subset k\beta \psi(\text{dist}(x, W))e - D.$$  

Therefore,

$$f(x') - f(x_0) + d \in k\beta \psi(\text{dist}(x, W))e - D.$$  

There is $d' \in D$ from (3.15) such that

$$f(x') - f(x_0) = k\beta \psi(\text{dist}(x, W))e - (d' + d).$$

Since $x' \in (S \cap U) \setminus W \subset \bigcup_{\lambda \in Q} S_1 \setminus W$, there is $\lambda' \in Q$ such that $x' \in S_{\lambda'}$. Moreover, $\Lambda \subset D'$ and $d + d' \in D$. Hence,

$$\langle \lambda', f(x') \rangle - \langle \lambda', f(x_0) \rangle = k\beta \psi(\text{dist}(x', W))\langle \lambda', e \rangle - \langle \lambda', d + d' \rangle \leq k\beta \psi(\text{dist}(x', W))\langle \lambda', e \rangle.$$  

By choosing $\beta = \beta_1^{-1}ak^{-1}$, we obtain a contradiction to (3.12).

(ii) Part “only if”: for each $\lambda \in Q$, we define,

$$S_\lambda = \{x \in S \cap U : \langle \lambda, f(x) \rangle \geq \langle \lambda, f(x_0) \rangle\}.$$  

Now, we will check that (3.6) holds true. Pick any $x \in S \cap U$. Suppose that $x \notin S_\lambda$ for any $\lambda \in Q$, then

$$\langle \lambda, f(x) - f(x_0) \rangle < 0, \quad \forall \lambda \in Q.$$  

(3.19)
Hence, for any $\lambda \in \text{cl} \text{co} Q = D^*$, $\langle \lambda, f(x) - f(x_0) \rangle \leq 0$. By applying the bipolar theorem, we have

$$f(x) - f(x_0) \in -D,$$

(3.20)

Combing it with the assumption, we have

$$f(x) - f(x_0) \in (-D) \cap D,$$

(3.21)

which is a contradiction to (3.19). So (3.6) holds and (3.3) is satisfied by the definition of $S_1$.

Part “if”: suppose that $x_0 \notin \text{Min}(f, S)$, then there exists $x \in S \cap U$ such that

$$f(x) - f(x_0) \in -D \setminus D.$$

(3.22)

Indeed, $x \in S \cap U$ can be replace by $x \in (S \cap U) \setminus W$, because $x \in W$, $f(x) - f(x_0) = 0$, which is contradiction to (3.22). Hence, for $x \in (S \cap U) \setminus W$, we have $\langle \lambda, f(x) - f(x_0) \rangle \leq 0$, $\forall \lambda \in D^*$. In particular,

$$\langle \lambda, f(x) - f(x_0) \rangle \leq 0, \quad \forall \lambda \in Q.$$

(3.23)

It follows from the assumption that

$$(\cup_{\lambda \in Q} S_1 \cap U) \setminus W \supset (S \cap U) \setminus W.$$

(3.24)

Therefore, by (3.3), we obtain

$$\langle \lambda, f(x) - f(x_0) \rangle > 0, \quad \forall \lambda \in Q, \quad \forall x \in (S_1 \cap U) \setminus W,$$

(3.25)

which contradicts relation (3.23).

$\square$

**Remark 3.2.** By taking $U = X$ in part (i) (resp., (ii)) of Theorem 3.1, we obtain a necessary and sufficient condition for $x_0$ to be in WS($\varphi, f, S$) (resp., Min($f$, S)). In particular, if we choose $Y = R^p$ and $D = R^p_+$ and $Q = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, then, we obtain Theorem 1 in [14].

Finally, we apply the nonlinear scalarization function to discuss the weak $\varphi$-sharp minimizer in vector optimization problems.

Let $D \subset Y$ be a closed and convex cone with nonempty interior int $D$. Given a fixed point $e \in \text{int} D$ and $y \in Y$, the nonlinear scalarization function $\xi : Y \to R$ is defined by

$$\xi(y) = \inf \{t : te \in y + D\}.$$

(3.26)

This function plays an important role in the context of nonconvex vector optimization problems and has excellent properties such as continuousness, convexity, and (strict) monotonicity on $Y$. More results about the function can be found in [17].
In what follows, we present several properties about the nonlinear scalarization function.

**Lemma 3.3** (see [17]). For any fixed $e \in \text{int } D$, $y \in Y$, and $r \in R$. One has

(i) $\xi(y) < r \iff re \in y + \text{int } D$,

(ii) $\xi(y) > r \iff re \notin y + D$,

(iii) $\xi(y) = r \iff re \in y + \text{bd } D$.

Given a vector-valued map $f : X \to Y$, define $\tilde{f} : X \to Y$ by

$$\tilde{f}(x) = f(x) - f(x_0).$$

(3.27)

Next, we consider weak $\psi$-sharp local minimizer for a vector-valued map $f$ through a weak sharp local minimizer of a scalar function $\xi \circ \tilde{f} : X \to R$.

**Theorem 3.4.** Let $x_0 \in S \subset X$. Suppose that $W$ defined by (2.4) is a closed set. Then,

$$x_0 \in \text{WSL}(\psi, f, S) \iff x_0 \in \text{WSL}(\psi, \xi \circ \tilde{f}, S).$$

(3.28)

**Proof.** Part “only if”: let us assume that $x_0 \in \text{WSL}(\psi, f, S)$. Thus, there exist $\alpha > 0$ and $U \in \mathcal{N}(x_0)$ such that

$$(f(x) - f(x_0) + D) \cap B(0, \alpha \psi(\text{dist}(x, W))) = \emptyset, \ \forall x \in (S \cap U) \setminus W.$$ 

(3.29)

Note that, when $W$ is a closed set,

$$\frac{\alpha}{4\|e\|} \psi(\text{dist}(x, W))e \in B(0, \alpha \psi(\text{dist}(x, W))) \ \forall x \in (S \cap U) \setminus W.$$ 

(3.30)

Therefore,

$$\frac{\alpha}{4\|e\|} \psi(\text{dist}(x, W))e \notin f(x) - f(x_0) + D \ \forall x \in (S \cap U) \setminus W.$$ 

(3.31)

By using Lemma 3.3(ii), one has

$$\xi(f(x) - f(x_0)) > \frac{\alpha}{4\|e\|} \psi(\text{dist}(x, W)) \ \forall x \in (S \cap U) \setminus W.$$ 

(3.32)

According to Lemma 3.3(iii), one has

$$\xi(f(x_0) - f(x_0)) = 0.$$ 

(3.33)
This relation, together with (3.32) yields
\[ \xi(f(x) - f(x_0)) > \xi(f(x_0) - f(x)) + \frac{\alpha}{4\|e\|}q_f(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W. \tag{3.34} \]

Namely,
\[ \left( \xi \circ \tilde{f} \right)(x) > \left( \xi \circ \tilde{f} \right)(x_0) + \frac{\alpha}{4\|e\|}q_f(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W, \tag{3.35} \]

that is, \( x_0 \in \text{WSL}(\psi, \xi \circ \tilde{f}, S) \).

Part “if”: by assumption, there exist \( \beta > 0 \) and \( U \in \mathcal{A}(x_0) \) such that
\[ \xi(\tilde{f}(x)) > \xi(\tilde{f}(x_0)) + \beta \psi(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W. \tag{3.36} \]

In terms of Lemma 3.3(iii), we have
\[ \xi(\tilde{f}(x_0)) = \xi(f(x_0) - f(x_0)) = 0. \tag{3.37} \]

Hence,
\[ \xi(f(x) - f(x_0)) > \beta \psi(\text{dist}(x,W)), \quad \forall x \in (S \cap U) \setminus W. \tag{3.38} \]

Once more using Lemma 3.3(ii), one has
\[ \beta \psi(\text{dist}(x,W)) e \notin f(x) - f(x_0) + D, \quad \forall x \in (S \cap U) \setminus W, \tag{3.39} \]

which implies that
\[ (\beta \psi(\text{dist}(x,W)) e - D) \cap (f(x) - f(x_0) + D) = \emptyset, \quad \forall x \in (S \cap U) \setminus W. \tag{3.40} \]

Since \( e \in \text{int} D \), there exists some number \( e > 0 \) such that \( B(0,e) \subset e - D \). Moreover,
\[ B(0,\lambda e) \subset \lambda e - D, \quad \forall \lambda > 0. \tag{3.41} \]

Hence, it follows from the relation that
\[ B(0,e\beta \psi(\text{dist}(x,W))) \subset \beta \psi(\text{dist}(x,W)) e - D, \quad \forall x \in (S \cap U) \setminus W. \tag{3.42} \]

Combing it with relation (3.40), we deduce that
\[ B(0,e\beta \psi(\text{dist}(x,W))) \cap (f(x) - f(x_0) + D) = \emptyset, \quad \forall x \in (S \cap U) \setminus W. \tag{3.43} \]
Let $\alpha = \epsilon \beta$, by the definition of weak $q$-sharp local minimizer, we have $x_0 \in \text{WS}(\varphi, f, S)$.

It is possible to illustrate Theorem 3.4 by means of adapting a simple example given in [14].

**Example 3.5.** Let $n = p = 2$, $S = \Omega = \mathbb{R}^2$, and $D = \mathbb{R}_+^3$ and let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

\[
\begin{align*}
  f_1(x^1, x^2) &:= \max \left\{ 0, \min \{ x^1, x^2 \} \right\} =
  \begin{cases} 
    x^1, & \text{if } x^2 \geq x^1 > 0, \\
    x^2, & \text{if } x^1 > x^2 > 0, \\
    0, & \text{if } x^1 \leq 0 \text{ or } x^2 \leq 0,
  \end{cases} \\

  f_2(x^1, x^2) &:= \max \left\{ 0, \min \{ -x^1, x^2 \} \right\} =
  \begin{cases} 
    -x^1, & \text{if } x^2 \geq -x^1 > 0, \\
    x^2, & \text{if } -x^1 > x^2 > 0, \\
    0, & \text{if } x^1 \geq 0 \text{ or } x^2 \leq 0,
  \end{cases}
\end{align*}
\]  

(3.44)

We choose $U = \mathbb{R}^2$. Using Definition 2.2, we derive that $x_0 = (0, 0) \in \text{WS}(\varphi_1, f, S)$.

Let $e = (1, 1)$. From Corollary 1.46 in [17], we have $(\xi \circ \tilde{f})(x) = \max_{1 \leq i \leq 2} f_i(x)$. Observe that

\[
W = \left\{ x : f(x) = (0, 0) \right\} = \left\{ x : x^2 \leq 0 \right\} \cup \left\{ x : x^1 = 0 \right\}.
\]  

(3.45)

It is easy to verify that $f_i(x) = \text{dist}(x, W)$ for all $x \in S \setminus W$. Using relation (2.7), we show that $x_0 = (0, 0) \in \text{WS}(\varphi_1, \xi \circ \tilde{f}, S)$. Hence, condition (3.28) with $q = q_1$ holds for $\alpha \in (0, 1)$.

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