Research Article

Approximate Fixed Point Theorems for the Class of Almost $S$-$KKM_C$ Mappings in Abstract Convex Uniform Spaces

Tong-Huei Chang, Chi-Ming Chen, and Yueh-Hung Huang

Department of Applied Mathematics, National Hsinchu University of Education, Hsin-Chu, Taiwan

Correspondence should be addressed to Chi-Ming Chen, ming@mail.nhcue.edu.tw

Received 25 February 2009; Accepted 11 June 2009

Recommended by Hichem Ben-El-Mechaiekh

We use a concept of abstract convexity to define the almost $S$-$KKM_C$ property, $al$-$S$-$KKM_C(X,Y)$ family, and almost $\Phi$-spaces. We get some new approximate fixed point theorems and fixed point theorems in almost $\Phi$-spaces. Our results extend some results of other authors.

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1. Introduction and Preliminaries

In 1929, Knaster et al. [1] proved the well-known $KKM$ theorem for an $n$-simplex. Ky Fan’s generalization of the $KKM$ theorem to infinite dimensional topological vector spaces in 1961 [2] proved to be a very versatile tool in modern nonlinear analysis with many far-reaching applications.


Among the many contributions in the study of the $KKM$ property and related topics, we mention the work by Amini et al. [5] where the classes of $KKM$ and $S$-$KKM$ mappings have been introduced in the framework of abstract convex spaces. The authors of [5] also define a concept of convexity that contains a number of other concepts of abstract convexities and obtain fixed point theorems for multifunctions verifying the $S$-$KKM$ property on $\Phi$-spaces that extend results of Ben-El-Mechaiekh et al. [6] and Horvath [7], motivated by the works of Ky Fan [2] and Browder [8]. We refer for the study of these notions to Ben-El-Mechaiekh et al. [9], and more recently, to Park [10], and Kim and Park [11].
In this paper, we use a concept of abstract convexity to define the almost $S$-\textit{KKM}$_{C}$ property, the corresponding notion of almost $S$-\textit{KKM}$_{C}(X,Y)$ family as well as the concept of almost $\Phi$-spaces.

Let $X$ and $Y$ be two sets, and let $T : X \to 2^{Y}$ be a set-valued mapping. We will use the following notations in the sequel:

(i) $T(x) = \{ y \in Y : y \in T(x) \}$,
(ii) $T(A) = \bigcup_{x \in A} T(x)$,
(iii) $T^{-1}(y) = \{ x \in X : y \in T(x) \}$,
(iv) $T^{-1}(B) = \{ x \in X : T(x) \cap B \neq \emptyset \}$, and
(v) if $D$ is a nonempty subset of $X$, then $(D)$ denotes the class of all nonempty finite subsets of $D$.

For the case where $X$ and $Y$ are two topological spaces, a set-valued map $T : X \to 2^{Y}$ is said to be closed if its graph $G_{T} = \{ (x,y) \in X \times Y : y \in T(x) \}$ is closed. $T$ is said to be compact if the image $T(X)$ of $X$ under $T$ is contained in a compact subset of $Y$.

\textit{Definition 1.1.} An abstract convex space $(E, C)$ consists of a nonempty topological space $E$, and a family $C$ of subsets of $E$ such that $E$ and $\emptyset$ belong to $C$ and $C$ is closed under arbitrary intersection. This kind of abstract convexity was widely studied; see [5, 9, 12, 13].

Suppose that $A$ is a nonempty subset of an abstract convex space $(E, C)$. Then

(i) a natural definition of $C$-convex hull of $A$ is

\[ co_{C}(A) = \cap \{ B \in C : A \subset B \} \quad (1.1) \]

(ii) we say that $A$ is $C$-convex if for each $B \in (A)$, $co_{C}(B) \subset A$.

\textit{Remark 1.2.} It is clear that if $A \in C$, then $A$ is $C$-convex. That is, each member of $C$ is $C$-convex.

\textit{Definition 1.3.} We list some properties of a uniform space. A uniformity [14] for a set $E$ is a nonempty family $\mathcal{U}$ of subsets of $E \times E$ such that

(i) each member of $\mathcal{U}$ contains the diagonal $\Delta$ where the diagonal $\Delta$ denotes the set of all pairs $(x,x)$ for $x$ in $E$;
(ii) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
(iii) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$;
(iv) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
(v) if $U \in \mathcal{U}$ and $U \subset V \subset E \times E$, then $V \in \mathcal{U}$.

The pair $(E, \mathcal{U})$ is called a uniform space. Every member $V$ in $\mathcal{U}$ is called an entourage. An entourage is said to be symmetric if $(x,y) \in V$ whenever $(y,x) \in V$.

\textit{Definition 1.4.} If $(E, C)$ is an abstract convex space with a uniformity $\mathcal{U}$, then we say that $(E, C)$ is an abstract convex uniform space.
Definition 1.5. Let $A$ be a nonempty subset of an abstract convex uniform space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$, and $\mathcal{U}$ has a symmetric basis $\mathcal{A}$. Then $A$ is called almost $C$-convex if, for any $K \in (A)$ and for any $V \in \mathcal{A}$, there exists a mapping $h_{K,V} : K \to A$ such that $h_{K,V}(x) \in V[x]$ for all $x \in K$ and $co_{C}(h_{K,V}(K)) \subset A$. Moreover, we call the mapping $h_{K,V} : K \to A$ a $C$-convex-inducing mapping.

Remark 1.6. It is clear that every $C$-convex set must be almost $C$-convex, but the converse is not true. And in general, the $C$-convex-inducing mapping is not unique. If $U, V \in \mathcal{A}$ and $U \subset V$, then $h_{A,U} : A \to X$ can be regarded as $h_{A,V} : A \to X$. If $A \subset B$, then $h_{A,U} : A \to X$ can be regarded as $h_{B,U} : B \to X$.

Recently, Amini et al. [5] introduced the class of multifunctions with the $S - KKM_{C}$ property in abstract convex spaces.

Definition 1.7 (see [5]). Let $Z$ be a nonempty set, $(X, \mathcal{C})$ an abstract convex space, and $Y$ a topological space. If $S : Z \to 2^{X}$, $T : X \to 2^{Y}$ and $F : Z \to 2^{Y}$ are three multifunctions satisfying

$$T(co_{C}(S(A))) \subset \cup_{x \in A} F(x), \quad \text{for each } A \in (Z),$$

then $F$ is called a $S$-$KKM_{C}$ mapping with respect to $T$. If the multifunction $T : X \to 2^{Y}$ satisfies the requirement that for any $S$-$KKM_{C}$ mapping $F$ with respect to $T$, the family $\{F(x) : x \in Z\}$ has the finite intersection property where $\overline{F(x)}$ denotes the closure of $F(x)$, then $T$ is said to have the $S$-$KKM_{C}$ property with respect to $C$. We define

$$S - KKM_{C}(Z, X, Y) := \left\{T : W \to 2^{Y} \mid T \text{ has the } S - KKM_{C} \text{ property with respect to } C\right\}.$$  

(1.3)

We extended the $S - KKM_{C}$ property to the almost $S - KKM_{C}$ property, as follows.

Definition 1.8. Let $Z$ be a nonempty set, let $X$ be an almost $C$-convex subset of an abstract convex uniform space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric basis $\mathcal{A}$, and let $Y$ be a topological space. If $S : Z \to 2^{X}$, $T : X \to 2^{Y}$ and $F : Z \to 2^{Y}$ are three multifunctions satisfying for each $A \in (Z)$, each $B \in (S(A))$, and each $U \in \mathcal{A}$, there exists a $C$-convex-inducing mapping $h_{B,U} : B \to W$ such that

$$T(co_{C}(h_{B,U}(B))) \subset F(A),$$

(1.4)

then $F$ is called an almost $S$-$KKM_{C}$ mapping with respect to $T$. If the multifunction $T : X \to 2^{Y}$ satisfies the requirement that for any almost $S$-$KKM_{C}$ mapping $F$ with respect to $T$, the family $\{\overline{F(x)} : x \in Z\}$ has the finite intersection property, then $T$ is said to have the almost $S$-$KKM_{C}$ property with respect to $C$. We define

$$al - S - KKM_{C}(Z, X, Y) := \left\{T : W \to 2^{Y} \mid T \text{ has the almost } S - KKM_{C} \text{ property with respect to } C\right\}.$$  

(1.5)
From the above definitions, we have the following proposition of the $al - S - KKM_C(Z, X, Y)$ family.

**Proposition 1.9.** Let $X$ be a nonempty set, let $Y$ be an almost $C$-convex subset of an abstract convex uniform space $(E, C)$, let $Z$ and $W$ be two topological spaces, and let $S : X \to 2^Y$ be a multifunction. If $T \in al - S - KKM_C(X, Y, Z)$ and if $f : Z \to W$ is continuous, then $fT \in al - S - KKM_C(X, Y, W)$.

The $\Phi$-mappings and the $\Phi$-spaces, in an abstract convex space setting, were also introduced by Amini et al. [5].

**Definition 1.10** (see [5]). Let $(X, C)$ be an abstract convex space, and $Y$ a topological space. A map $T : Y \to 2^X$ is called a $\Phi$-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ implies $co_C(A) \subseteq T(y)$, and

(ii) $Y = \bigcup_{x \in X} int F^{-1}(x)$.

The mapping $F$ is called a companion mapping of $T$.

Furthermore, if the abstract convex space $(X, C)$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric basis family $\mathcal{A}$, then $X$ is called a $\Phi$-space if for each entourage $V \in \mathcal{A}$, there exists a $\Phi$-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subseteq V$.

**Remark 1.11.**

(i) If $T : Y \to 2^X$ is a $\Phi$-mapping, then for each nonempty subset $Y_1$ of $Y$, $T|_{Y_1} : Y_1 \to X$ is also a $\Phi$-mapping.

(ii) It is easy to see that if $X_1 \subset X$ and $C_1 = \{C \cap X_1 : C \in C\}$, then $(X_1, C_1)$ is also a $\Phi$-space.

In order to establish the main result of this paper for the multifunctions with the almost $S - KKM_C$ property, we need the following definitions concerning the almost $\Phi$-mappings and the almost $\Phi$-spaces.

**Definition 1.12.** Let $X$ be an almost $C$-convex subset of an abstract convex uniform space $(E, C)$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric base family $\mathcal{A}$, and $Y$ a topological space. A map $T : Y \to 2^X$ is called an almost $\Phi$-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ and $U \in \mathcal{A}$, there exists a $C$-convex-inducing $h_{AU} : A \to X$ such that $co_C(h_{AU}(A)) \subseteq U[T(y)]$, and

(ii) $Y = \bigcup_{x \in X} int F^{-1}(x)$.

The mapping $F$ is called an almost companion mapping of $T$.

Furthermore, $X$ is called an almost $\Phi$-space, if, for each entourage $V \in \mathcal{A}$, there exists an almost $\Phi$-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subseteq V$.

**Definition 1.13.** Let $X$ be an almost $\Phi$-space, and let $T : X \to 2^X$. We say that $T$ has the approximate fixed point property if, for each $U \in \mathcal{A}$, there exists $x \in X$ such that $U[x] \cap T(x) \neq \emptyset$.

### 2. Main Results

Using the above introduced concepts and definitions, we now state our main theorem.
Theorem 2.1. Let $X$ be an almost $\Phi$-space, and let $s : X \to X$ be a surjective single-valued function. If $T \in \sigma - s - KKM_C(X,X,X)$ is compact, then $T$ has the approximate fixed point property.

Proof. Let $\mathcal{N}$ be a symmetric basis of the uniform structure, and let $U \in \mathcal{N}$. Take $V \in \mathcal{N}$ such that $V \circ V \subset U$. Then, by the definition of the almost $\Phi$-space, there exists an almost $\Phi$-mapping $F : X \to 2^X$ such that $G_F \subset V$. Since $F$ is an almost $\Phi$-mapping, there exists an almost companion mapping $G : X \to 2^X$ such that $X = \cup_{x \in X} \text{int} G^{-1}(x)$.

Let $K = \overline{T(X)}$. Then $K$ is compact, since $T$ is compact. Hence there exists $A \in \langle X \rangle$ such that $K \subset \cup_{x \in A} \text{int} G^{-1}(x)$. Since $s$ is surjective, there exists a finite subset $B$ of $X$ such that $K \subset \cup_{x \in B} \text{int} G^{-1}(s(x))$.

Now, we define $P : X \to 2^X$ by

$$P(z) = K \setminus \text{int} G^{-1}(s(z)), \text{ for each } z \in X. \quad (2.1)$$

By the definition of $P$, we obtain that $P$ is not an almost $s - KKM_C$ mapping with respect to $T$. Hence, there exist $N = \{z_1, z_2, \ldots, z_k\} \subset X$ and $D \in \langle s(N) \rangle$ such that for any $C$-convex-inducing $h_{D,V} : D \to W_{z_i}$, we have

$$T(\text{co}_C(h_{D,V}(D))) \not\subset \cup_{i=1}^k P(z_i). \quad (2.2)$$

So, for any $C$-convex-inducing $h_{D,V} : D \to X$, there exist $x_U \in \text{co}_C(h_{D,V}(D))$ and $y_U \in T(x_U)$ such that $y_U \notin \cup_{i=1}^k P(z_i)$. Consequently, $y_U \in \cap_{i=1}^k \text{int} G^{-1}(s(z_i))$, and so $s(z_i) \in G(y_U)$ for all $i = 1, 2, \ldots, k$. Since $F$ is an almost $\Phi$-mapping, there exists a $C$-convex-inducing $h_{D,V}^* : D \to X$ such that $\text{co}_C(h_{D,V}^*(D)) \subset V[F(y_U)]$. So $x_U \in \text{ad}_C(h_{D,V}^*(D))$ and $x_U \in V[F(y_U)]$. Thus, there exists $z_U \in F(y_U)$ such that $x_U \in V[z_U]$. Since $X$ is an almost $\Phi$-space, we have $(y_U, z_U) \in G_F \subset V$, and so $(y_U, x_U) = (y_U, z_U) \circ (z_U, x_U) \in V \circ V \subset U$, that is, $y_U \in U[x_U]$. Therefore, $y_U \in U[x_U] \cap T(x_U)$. The proof is finished.

Remark 2.2. In the case, if $X$ is a $\Phi$-space and $T \in \sigma - KKM_C(X, X, X)$, then the above theorem reduces to Amini et al. [5, Theorem 2.5]

From Theorem 2.1 above, we obtain immediately the following fixed point theorem.

Theorem 2.3. Suppose that all of the assumptions of Theorem 2.1 hold. If $T$ is closed, then $T$ has a fixed point in $X$.

Proof. By Theorem 2.1, for each $U \in \mathcal{N}$, there exist $x_U, y_U \in X$ such that $y_U \in U[x_U] \cap T(x_U)$. Since $T$ is compact, without loss of generality, we may assume that $y_U$ converges to some $\overline{y}$ in $X$; then $x_U$ also converges to $\overline{y}$ since $X$ is a Hausdorff uniform space and $(x_U, y_U) \in U$ for each $U \in \mathcal{N}$. By the closedness of $T$, we have that $\overline{y} \in T(\overline{y})$.

Corollary 2.4. Let $X$ be an almost $\Phi$-space, and let $s : X \to X$ be a surjective single-valued function. Suppose $T \in \sigma - s - KKM_C(X, X, X)$ such that $\overline{T(X)}$ is totally bounded. Then $T$ has the approximate fixed point property.

Corollary 2.5. Suppose that all of the assumptions of the above Corollary 2.5 hold. If $T$ is closed, then $T$ has a fixed point in $X$. 
In case $X$ is an almost convex subset of Hausdorff topological vector spaces and for each $A \subset X$, we have

(i) $\text{co}_C(A) = \text{co}(A)$, and

(ii) $al - s - KKM_C(X, X, X) = al - s - KKM(X, X, X)$.

This allows us to state the following results.

**Theorem 2.6.** Let $E$ be a Hausdorff locally convex space, let $X$ be an almost convex subset of $E$, and let $s : X \to X$ be a surjective function. Assume that $T \in al - s - KKM(X, X, X)$ is compact and closed, then $T$ has a fixed point in $X$.

**Proof.** Let $C$ be the family of all convex subsets of $E$, and let $B_0 = \{ \overline{V}_a : \alpha \in \Lambda \}$ be a local basis of $E$ such that each $\overline{V}_a \in B_0$ is symmetric and convex for each $\alpha \in \Lambda$. For each $x \in X$, we set $V_a[x] = x + \overline{V}_a$. Noting that $x \in V_a[x]$. Set

$$\mathcal{A} = \{ V_a | V_a = \cup_{x \in X} \{ (x, y) : y \in V_a[x] \}, \alpha \in \Lambda \}. \quad (2.3)$$

Then $\mathcal{A}$ is a basis of a uniformity of $E$. For each $V_\beta \in \mathcal{A}$, $\beta \in \Lambda$, we define the two set-valued mappings $G, F : X \to 2^X$ by $G(x) = F(x) = V_\beta[x]$ for each $x \in X$. Then we have

(i) for each $x \in X$, $\text{co}(G(x)) = \text{co}(V_\beta[x]) \subset V_\beta[V_\beta[x]] = V_\beta[F(x)]$, and

(ii) $X = \cup_{x \in X} \text{int} G^{-1}(x)$.

So, $G$ is an almost companion mapping of $F$. This implies that $F$ is an almost $\Phi$-mapping such that $G_F \subset V_\beta$. Therefore, $X$ is an almost $\Phi$-space.

All conditions of Theorems 2.1 and 2.3 are therefore fulfilled; the result follows from an argument similar to those in the proofs of Theorems 2.1 and 2.3. □

**Theorem 2.7.** Let $E$ be a topological vector space, let $X$ be an almost convex subset of $E$, and let $s : X \to X$ be a surjective function. Suppose that $T \in al - s - KKM(X, X, X)$ is compact, then for any symmetric convex neighborhood $\overline{V}$ of 0 in $E$, there is $x_\overline{V} \in X$ such that $(x_\overline{V} + \overline{V}) \cap T(x_\overline{V}) \neq \emptyset$.

**Proof.** Let $C$ be the family of all convex subsets of $E$, and let $B_0 = \{ a\overline{V} : a > 0 \}$ be a new local basis of $E$. We will use $B_0$ to construct a weaker topology on $E$ such that $E$ becomes a new topological vector space. For each $x \in X$, we set $V_a[x] = x + a\overline{V}$. Noting that $x \in V_a[x]$. Set

$$\mathcal{A} = \{ V_a | V_a = \cup_{x \in X} \{ (x, y) : y \in V_a[x] \}, \ a > 0 \}. \quad (2.4)$$

Then $\mathcal{A}$ is a basis of a uniformity of $E$. In vein of the reasonings similar to those of Theorems 2.1 and 2.6, we complete the proof. □
References