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Research Article

The Alexandroff-Urysohn Square and the Fixed Point Property

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Every continuous function of the Alexandroff-Urysohn Square into itself has a fixed point. This follows from G. S. Young's general theorem (1946) that establishes the fixed-point property for every arcwise connected Hausdorff space in which each monotone increasing sequence of arcs is contained in an arc. Here we give a short proof based on the structure of the Alexandroff-Urysohn Square.

Alexandroff and Urysohn [1] in *Mémoire sur les espaces topologiques compacts* defined a variety of important examples in general topology. The final manuscript for this classical paper was prepared in 1923 by Alexandroff shortly after the death of Urysohn. On [1, page 15], Alexandroff denoted a certain space by U_1. While Steen and Seebach in *Counterexamples in Topology* [2, Example 101] refer to this space as the Alexandroff Square, we concur with Cameron [3, pages 791-792], who attributes it to Urysohn. Hence we refer to U_1 as the Alexandroff-Urysohn Square and for convenience denote it by (X, τ). The following definition of (X, τ) is given by Steen and Seebach [2, Example 101, pages 120-121]. Define X to be the closed unit square $[0,1]\times[0,1]$ with the topology τ defined by taking as a neighborhood basis of each point (s,t) off the diagonal $\Delta = \{(x,x) \in X \mid x \in [0,1]\}$ the intersection of $X \setminus \Delta$ with open vertical line segments centered at (s,t) (e.g., $N_\epsilon(s,t) = \{(s,y) \in X \setminus \Delta \mid |t-y| < \epsilon\}$). Neighborhoods of each point $(s,s) \in \Delta$ are the intersection with X of open horizontal strips less a finite number of vertical lines (e.g., $M_\epsilon(s,s) = \{(x,y) \in X \mid |y-s| < \epsilon$ and $x \neq x_0, x_1, \ldots, x_n\}$). Note (X, τ) is not first countable, and therefore not metrizable. However, (X, τ) is a compact arcwise-connected Hausdorff space [2].

In Young's paper [4] of 1946, local connectivity is introduced on a space by a change of topology with consequent implications on generalized dendrites. A non-specialist may not notice that the fixed-point property for the Alexandroff-Urysohn Square follows from a result in Young's paper. We offer the following short proof based on the structure of the Alexandroff-Urysohn Square. The proof is direct and uses a dog-chases-rabbit argument [5, page 123–125]; first having the dog run up the diagonal, and then up (or down) a vertical fiber. The Alexandroff-Urysohn Square is a Hausdorff dendroid. For a dog-chases-rabbit argument that metric dendroids have the fixed point property, see [6], and also see [7].

Definition 1. A set U in (X,τ) is an *ordered segment* if U is a connected vertical linear neighborhood or U is a component of the intersection of Δ and a horizontal strip neighborhood.

Note the relative topology induced on each ordered segment by τ is the Euclidean topology. Each point of (X,τ) is contained in arbitrarily small ordered segments.

Let $\pi_1 : (X,\tau) \to [0,1]$ be the function defined by $\pi_1(x,y) = x$. Since each neighborhood in (X,τ) of a point of Δ is projected by π_1 onto the complement of a finite set in $[0,1]$, the function π_1 is discontinuous at each point of Δ.

Let $\pi_2 : (X,\tau) \to [0,1]$ be the function defined by $\pi_2(x,y) = y$. Note π_2 is continuous.

Lemma 2. *Let $f : (X,\tau) \to (X,\tau)$ be a continuous function. Let $p = (x,x)$ be a point of Δ. If $\pi_1 f(p) \neq x$, then there is an ordered segment U containing p such that $\pi_1 f(U)$ is in one component of $[0,1] \setminus \pi_1(U)$.*

Proof. Suppose $\pi_1 f(p) \neq x$. We consider two cases.

Case 1. Assume $f(p) \notin \Delta$. Let V be a vertical ordered segment containing $f(p)$.

Since $p \in \Delta$ and f is continuous, there is a horizontal strip neighborhood H in (X,τ) of p such that $\pi_1(V) \notin \pi_1(H \cap \Delta)$ and $f(H) \subset V$. Let U be the p-component of $H \cap \Delta$. Note U is an ordered segment containing p and $f(U) \subset V$. The point $\pi_1 f(U)$ is contained in one component of $[0,1] \setminus \pi_1(U)$.

Case 2. Assume $f(p) \in \Delta$. Let K be a horizontal strip neighborhood in (X,τ) of $f(p)$ such that $x \notin \pi_1(K \cap \Delta)$ and $K \cap \Delta$ is connected. Let L be the $f(p)$-component of K. Note L is a square set with diagonal $K \cap \Delta$.

Let H be a horizontal strip neighborhood in (X,τ) of p such that $H \cap K = \emptyset$ and $f(H) \subset K$. Let U be the ordered segment that is the p-component of $H \cap \Delta$. Note $f(U)$ is a connected subset of L and $\pi_1(U) \cap \pi_1(L) = \emptyset$. Hence $\pi_1 f(U)$ is in one component of $[0,1] \setminus \pi_1(U)$. This completes the proof of our lemma. □

Theorem 3. *The Alexandroff-Urysohn Square (X,τ) has the fixed-point property.*

Proof. Let $f : (X,\tau) \to (X,\tau)$ be a continuous function. We will show there exists a point of (X,τ) that is not moved by f.

Let $B = \{x \in [0,1] \mid \pi_1 f(x,x) \geq x\}$. Note $0 \in B$. Let b be the least upper bound of B.

Note $\pi_1 f(b,b) = b$. To see this assume $\pi_1 f(b,b) \neq b$. Then, by the lemma, there is an ordered segment U in Δ containing (b,b) such that $\pi_1 f(U)$ is in one component of $[0,1] \setminus \pi_1(U)$. However since b is the least upper bound of B, there exist points a and c in $\pi_1(U)$ such that $\pi_1 f(a,a) \geq a$ and $\pi_1 f(c,c) < c$, a contradiction. Hence, $\pi_1 f(b,b) = b$.

If $\pi_2 f(b,b) = b$, then $f(b,b) = (b,b)$ as desired.

If $\pi_2 f(b,b) \neq b$, then either $\pi_2 f(b,b) > b$ or $\pi_2 f(b,b) < b$. Assume without loss of generality that $\pi_2 f(b,b) > b$.

Let I denote the interval $\{b\} \times [b,1]$.

Let $r : (X,\tau) \to (X,\tau)$ be the function defined by $r(p) = p$ if $p \in I$ and $r(p) = (b,b)$ if $p \notin I$.

Note $\{b\} \times (b,1]$ is an open and closed subset of $X \setminus \{(b,b)\}$. It follows that r is continuous. Thus, r is a retraction of (X,τ) to I.

Let $\widehat{f}$ be the restriction of f to I. Since $r\widehat{f}$ is a continuous function of the interval I into itself, there is a point $(b,d) \in I$ such that $r\widehat{f}(b,d) = (b,d)$.

Since every point of I that is sent into $X \setminus I$ by f is moved by $r\widehat{f}$, it follows that $f(b,d) \in I$. Hence $f(b,d) = r\widehat{f}(b,d) = (b,d)$. □

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