

Research Article

Some Fixed Point Theorem for Mapping on Complete G -Metric Spaces

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We prove some fixed point results for mapping satisfying sufficient conditions on complete G -metric space, also we showed that if the G -metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space (X, d_G) , where (X, d_G) is the usual metric space which defined from the G -metric space (X, G) .

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1. Introduction

During the sixties, the notion of 2-metric space introduced by Gähler (see [1, 2]) as a generalization of usual notion of metric space (X, d) . But different authors proved that there is no relation between these two functions, for instance, Ha et al. in [3] show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D. thesis introduce a new class of generalized metric space called D -metric spaces ([4, 5]).

In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [5–7]). He claimed that D -metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

But in 2003 in collaboration with Brailey Sims, we demonstrated in [8] that most of the claims concerning the fundamental topological structure of D -metric space are incorrect, so, we introduced more appropriate notion of generalized metric space as follows.

Definition 1.1 (see [9]). Let X be a nonempty set, and let $G : X \times X \times X \rightarrow \mathbf{R}^+$ be a function satisfying the following properties:

$$(G1) \quad G(x, y, z) = 0 \text{ if } x = y = z;$$

$$(G2) \quad 0 < G(x, x, y); \text{ for all } x, y \in X, \text{ with } x \neq y;$$

- (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;
 (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables);
 (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically, a *G-metric* on X , and the pair (X, G) is called a *G-metric space*.

Definition 1.2 (see [9]). Let (X, G) be a G-metric space, and let (x_n) be sequence of points of X , a point $x \in X$ is said to be the *limit* of the sequence (x_n) , if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is *G-convergent* to x .

Thus, that if $x_n \rightarrow x$ in a G-metric space (X, G) , then for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.3 (see [9]). *Let (X, G) be a G-metric space, then the following are equivalent.*

- (1) (x_n) is G-convergent to x .
- (2) $G(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.
- (3) $G(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$.
- (4) $G(x_m, x_n, x) \rightarrow 0$, as $m, n \rightarrow \infty$.

Definition 1.4 (see [9]). Let (X, G) be a G-metric space, a sequence (x_n) is called *G-Cauchy* if for every $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Proposition 1.5 (see [8]). *If (X, G) is a G-metric space, then the following are equivalent.*

- (1) *The sequence (x_n) is G-Cauchy.*
- (2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 1.6 (see [9]). Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \rightarrow (X', G')$ be a function, then f is said to be *G-continuous at a point $a \in X$* if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is *G-continuous at X* if and only if it is G-continuous at all $a \in X$.

Proposition 1.7 (see [9]). *Let (X, G) , (X', G') be two G-metric spaces. Then a function $f : X \rightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x ; that is, whenever (x_n) is G-convergent to x , $(f(x_n))$ is G-convergent to $f(x)$.*

Definition 1.8 (see [9]). A G-metric space (X, G) is called *symmetric G-metric space* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.9 (see [9]). *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.10 (see [8]). *Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \quad (1.1)$$

Note that if (X, G) is a symmetric G-metric space, then

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \quad (1.2)$$

However, if (X, G) is not symmetric, then it holds by the G -metric properties that

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X, \quad (1.3)$$

and that in general these inequalities cannot be improved.

Definition 1.11 (see [9]). A G -metric space (X, G) is said to be G -complete (or complete G -metric) if every G -Cauchy sequence in (X, G) is G -convergent in (X, G) .

Proposition 1.12 (see [9]). A G -metric space (X, G) is G -complete if and only if (X, d_G) is a complete metric space.

2. Main results

Here we start our work with the following theorem.

Theorem 2.1. Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:

$$G(T(x), T(y), T(z)) \leq \{aG(x, y, z) + bG(x, T(x), T(x)) + cG(y, T(y), T(y)) + dG(z, T(z), T(z))\} \quad (2.1)$$

or

$$G(T(x), T(y), T(z)) \leq \{aG(x, y, z) + bG(x, x, T(x)) + cG(y, y, T(y)) + dG(z, z, T(z))\} \quad (2.2)$$

for all $x, y, z \in X$ where $0 \leq a + b + c + d < 1$, then T has a unique fixed point (say u , i.e., $Tu = u$), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.1), then for all $x, y \in X$, we have

$$\begin{aligned} G(Tx, Ty, Ty) &\leq aG(x, y, y) + bG(x, Tx, Tx) + (c + d)G(y, Ty, Ty), \\ G(Ty, Tx, Tx) &\leq aG(y, x, x) + bG(y, Ty, Ty) + (c + d)G(x, Tx, Tx). \end{aligned} \quad (2.3)$$

Suppose that (X, G) is symmetric, then by definition of metric (X, d_G) and (1.2), we get

$$d_G(Tx, Ty) \leq ad_G(x, y) + \frac{c + d + b}{2}d_G(x, Tx) + \frac{c + d + b}{2}d_G(y, Ty), \quad \forall x, y \in X. \quad (2.4)$$

In this line, since $0 < a + b + c + d < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in metric space (X, d_G) (see [10]).

However, if (X, G) is not symmetric then by definition of metric (X, d_G) and (1.3), we get

$$d_G(Tx, Ty) \leq ad_G(x, y) + \frac{2(c + d + b)}{3}d_G(x, Tx) + \frac{2(c + d + b)}{3}d_G(y, Ty), \quad (2.5)$$

for all $x, y \in X$, then the metric condition gives no information about this map since $0 < a + 2(c + d + b)/3 + 2(c + d + b)/3$ need not be less than 1. But this can be proved by G -metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (2.1), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + (c+d)G(x_n, x_{n+1}, x_{n+1}), \quad (2.6)$$

then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)}G(x_{n-1}, x_n, x_n). \quad (2.7)$$

Let $q = (a+b)/(1-(c+d))$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n). \quad (2.8)$$

Continuing in the same argument, we will get

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \quad (2.9)$$

Moreover, for all $n, m \in \mathbf{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q}G(x_0, x_1, x_1), \end{aligned} \quad (2.10)$$

and so $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$. Thus (x_n) is G -Cauchy sequence. Due to the completeness of (X, G) , there exists $u \in X$ such that (x_n) is G -converge to u .

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq aG(x_{n-1}, u, u) + bG(x_{n-1}, x_n, x_n) + (c+d)G(u, T(u), T(u)), \quad (2.11)$$

taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, then $G(u, T(u), T(u)) \leq (c+d)G(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then

$$G(u, v, v) \leq aG(u, v, v) + bG(u, T(u), T(u)) + (c+d)G(v, T(v), T(v)) = aG(u, v, v), \quad (2.12)$$

which implies that $u = v$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$. we can deduce that

$$\begin{aligned} G(u, T(y_n), T(y_n)) &\leq aG(u, y_n, y_n) + bG(u, T(u), T(u)) + (c+d)G(y_n, T(y_n), T(y_n)) \\ &= aG(u, y_n, y_n) + (c+d)G(y_n, T(y_n), T(y_n)), \end{aligned} \quad (2.13)$$

and since $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, we have that $G(u, T(y_n), T(y_n)) \leq (a/(1-(c+d)))G(u, y_n, y_n) + ((c+d)/(1-(c+d)))G(y_n, u, u)$.

Taking the limit as $n \rightarrow \infty$, from which we see that $G(u, T(y_n), T(y_n)) \rightarrow 0$ and so, by Proposition 1.7, $T(y_n) \rightarrow u = Tu$. It is proved that T is G -continuous at u .

If T satisfies condition (2.2), then the argument is similar to that above. However, to show that the sequence (x_n) is G -Cauchy, we start with

$$G(x_n, x_n, x_{n+1}) \leq aG(x_{n-1}, x_{n-1}, x_n) + (b+c)G(x_{n-1}, x_{n-1}, x_n) + dG(x_n, x_n, x_{n+1}), \quad (2.14)$$

then

$$G(x_n, x_n, x_{n+1}) \leq \frac{a+b+c}{1-d}G(x_{n-1}, x_{n-1}, x_n). \quad (2.15)$$

Let $q = (a+b+c)/(1-d)$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

Continuing in the same way, we find that

$$G(x_n, x_n, x_{n+1}) \leq q^n G(x_0, x_0, x_1). \quad (2.16)$$

Then for all $n, m \in \mathbf{N}$; $n < m$, we have by repeated use of the rectangle inequality $G(x_n, x_n, x_m) \leq (q^n/(1-q))G(x_0, x_0, x_1)$. \square

Corollary 2.2. *Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:*

$$\begin{aligned} &G(T^m(x), T^m(y), T^m(z)) \\ &\leq \{aG(x, y, y) + bG(x, T^m(x), T^m(x)) + cG(y, T^m(y), T^m(y)) + dG(z, T^m(z), T^m(z))\} \end{aligned} \quad (2.17)$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq \{aG(x, y, y) + bG(x, x, T^m(x)) + cG(y, y, T^m(y)) + dG(z, z, T^m(z))\}, \quad (2.18)$$

for all $x, y, z \in X$, where $0 \leq a+b+c+d < 1$. Then T has a unique fixed point (say u), and T^m is G -continuous at u .

Proof. From the previous theorem, we see that T^m has a unique fixed point (say u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. \square

Theorem 2.3. *Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{l} G(x, T(x), T(x)), \\ G(y, T(y), T(y)), \\ G(z, T(z), T(z)) \end{array} \right\} \quad (2.19)$$

or

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{l} G(x, x, T(x)), \\ G(y, y, T(y)), \\ G(z, z, T(z)) \end{array} \right\}, \quad (2.20)$$

for all $x, y, z \in X$, where $0 \leq k < 1$. Then T has a unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.19), then for all $x, y \in X$,

$$\begin{aligned} G(Tx, Ty, Ty) &\leq k \max\{G(x, Tx, Tx), G(y, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq k \max\{G(y, Ty, Ty), G(x, Tx, Tx)\}. \end{aligned} \quad (2.21)$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2) we get

$$d_G(Tx, Ty) \leq k \max\{d_G(x, Tx), d_G(y, Ty)\}, \quad \forall x, y \in X. \quad (2.22)$$

Since $k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G) (see [11]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we get

$$d_G(Tx, Ty) \leq \frac{4k}{3} \max\{d_G(x, Tx), d_G(y, Ty)\}, \quad \forall x, y \in X. \quad (2.23)$$

The metric condition gives no information about this map since $4k/3$ need not be less than 1, but we will proof it by G -metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (2.19), we can verify that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq k \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &= kG(x_{n-1}, x_n, x_n) \quad (\text{since } 0 \leq k < 1). \end{aligned} \quad (2.24)$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1). \quad (2.25)$$

For all $n, m \in \mathbb{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1})G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned} \quad (2.26)$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$, and thus (x_n) is G -Cauchy sequence. Due to the completeness of (X, G) , there exists $u \in X$ such that $(x_n) \rightarrow u$.

Suppose that $T(u) \neq u$, then $G(x_{n+1}, T(u), T(u)) \leq k \max\{G(x_{n+1}, x_{n+2}, x_{n+2}), G(u, T(u), T(u))\}$ and by taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, we get that $G(u, T(u), T(u)) \leq kG(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then $G(u, v, v) \leq k \max\{G(v, v, v), G(u, u, u)\} = 0$ which implies that $u = v$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq k \max\{G(u, T(u), T(u)), G(y_n, T(y_n), T(y_n))\} = kG(y_n, T(y_n), T(y_n)). \quad (2.27)$$

But, $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, then $G(u, T(y_n), T(y_n)) \leq (k/(1-k))G(y_n, u, u)$. Taking the limit as $n \rightarrow \infty$, from which we see that $G(u, T(y_n), T(y_n)) \rightarrow 0$, and so by Proposition 1.7, $T(y_n) \rightarrow u = Tu$. So, T is G -continuous at u \square

Corollary 2.4. Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions for some $m \in \mathbb{N}$:

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{l} G(x, T^m(x), T^m(x)), \\ G(y, T^m(y), T^m(y)), \\ G(z, T^m(z), T^m(z)) \end{array} \right\} \quad (2.28)$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{l} G(x, x, T^m(x)), \\ G(y, y, T^m(y)), \\ G(z, z, T^m(z)) \end{array} \right\}, \quad (2.29)$$

for all $x, y, z \in X$, then T has a unique fixed point (say u) and T^m is G -continuous at u .

Proof. We use the same argument as in Corollary 2.2. \square

Theorem 2.5. Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, T(y), T(y)), G(y, T(x), T(x)), G(y, T(y), T(y))\} \quad (2.30)$$

or

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, x, T(y)), G(y, y, T(x)), G(y, y, T(y))\}, \quad (2.31)$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.30), then for all $x, y \in X$,

$$\begin{aligned} G(Tx, Ty, Ty) &\leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(y, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq k \max\{G(x, Ty, Ty), G(y, Tx, Tx), G(x, Tx, Tx)\}. \end{aligned} \quad (2.32)$$

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2), we have.

$$\begin{aligned} d_G(Tx, Ty) &\leq \frac{k}{2} \max \left\{ \begin{array}{l} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(y, Ty) \end{array} \right\} + \frac{k}{2} \max \left\{ \begin{array}{l} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(x, Tx) \end{array} \right\} \\ &\leq k \max\{d_G(x, Ty), d_G(y, Tx), d_G(x, Tx), d_G(y, Ty)\}, \quad \forall x, y \in X. \end{aligned} \quad (2.33)$$

Since $0 \leq k < 1$, then the existence and uniqueness of the fixed point follows from a theorem in metric space (X, d_G) (see [12]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we have

$$d_G(Tx, Ty) \leq \frac{2k}{3} \max \left\{ \begin{array}{l} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(y, Ty) \end{array} \right\} + \frac{2k}{3} \max \left\{ \begin{array}{l} d_G(x, Ty), \\ d_G(y, Tx), \\ d_G(x, Tx) \end{array} \right\}, \quad (2.34)$$

for all $x, y \in X$, then the metric space (X, d_G) gives no information about this map since $4k/3$ need not be less than 1. But we will proof it by G -metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.30) and using $k < 1$, we deduce that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} = kG(x_{n-1}, x_{n+1}, x_{n+1}). \quad (2.35)$$

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_{n+1}, x_{n+1}), \quad (2.36)$$

and using

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq k \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1}), G(x_n, x_{n+1}, x_{n+1})\}, \quad (2.37)$$

then,

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^2 \max\{G(x_{n-2}, x_{n+1}, x_{n+1}), G(x_n, x_{n-1}, x_{n-1})\}. \quad (2.38)$$

Continuing in this procedure, we will have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n \Gamma_n, \quad (2.39)$$

where $\Gamma_n = \max\{G(x_i, x_j, x_j); \text{ for all } i, j \in \{0, 1, \dots, n+1\}\}$.

For $n, m \in \mathbf{N}; n < m$, let $\Gamma = \max\{\Gamma_i; \text{ for all } i = n, \dots, m-1\}$.

Then, for all $n, m \in \mathbf{N}; n < m$, we have by rectangle inequality

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \\ &\leq k^n \Gamma_n + k^{n+1} \Gamma_{n+1} + \dots + k^{m-1} \Gamma_{m-1} \\ &\leq (k^n + k^{n+1} + \dots + k^{m-1}) \Gamma \\ &\leq \frac{k^n}{1-k} \Gamma. \end{aligned} \quad (2.40)$$

This prove that $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$, and thus (x_n) is G -Cauchy sequence. Since (X, G) is G -complete then there exists $u \in X$ such that (x_n) is G -converge to u .

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq k \max\{G(x_{n-1}, T(u), T(u)), G(u, x_{n+1}, x_{n+1}), G(u, T(u), T(u))\}. \quad (2.41)$$

Taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, we get $G(u, T(u), T(u)) \leq kG(u, T(u), T(u))$, this contradiction implies that $u = T(u)$.

To prove the uniqueness, suppose that $u \neq v$ such that $T(v) = v$. So, by (2.30), we have that

$$G(u, v, v) \leq k \max\{G(u, v, v), G(v, u, u)\} \implies G(u, v, v) \leq kG(v, u, u). \quad (2.42)$$

Again we will find $G(v, u, u) \leq kG(u, v, v)$, so

$$G(u, v, v) \leq k^2 G(u, v, v); \quad (2.43)$$

since $k < 1$, this implies that $u = v$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq k \max\{G(u, T(y_n), T(y_n)), G(y_n, T(u), T(u)), G(y_n, T(y_n), T(y_n))\}. \quad (2.44)$$

But, $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, so, $G(u, T(y_n), T(y_n)) \leq (k/(1-k))G(y_n, u, u)$.

Taking the limit as $n \rightarrow \infty$, from which we see that $G(u, T(y_n), T(y_n)) \rightarrow 0$ and so, by Proposition 1.7, we have $T(y_n) \rightarrow u = Tu$ which implies that T is G -continuous at u . \square

Corollary 2.6. *Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$, be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{l} G(x, T(y), T(y)), G(x, T(z), T(z)), \\ G(y, T(x), T(x)), G(y, T(z), T(z)), \\ G(z, T(x), T(x)), G(z, T(y), T(y)) \end{array} \right\} \quad (2.45)$$

or

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{l} G(x, x, T(y)), G(x, x, T(z)), \\ G(y, y, T(x)), G(y, y, T(z)), \\ G(z, z, T(x)), G(z, z, T(y)) \end{array} \right\}, \quad (2.46)$$

for all $x, y, z \in X$ where $k \in [0, 1)$, then T has a unique fixed point (say u) and T is G -continuous at u .

Proof. If we let $z = y$ in conditions (2.45) and (2.46), then they become conditions (2.30) and (2.31), respectively, in Theorem 2.5; so the proof follows from Theorem 2.5. \square

Corollary 2.7. *Let (X, G) be a complete G -metric space and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:*

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{l} G(x, T^m(y), T^m(y)), G(x, T^m(z), T^m(z)), \\ G(y, T^m(x), T^m(x)), G(y, T^m(z), T^m(z)), \\ G(z, T^m(x), T^m(x)), G(z, T^m(y), T^m(y)) \end{array} \right\},$$

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{l} G(x, x, T^m(y)), G(x, x, T^m(z)), \\ G(y, y, T^m(x)), G(y, y, T^m(z)), \\ G(z, z, T^m(x)), G(z, z, T^m(y)) \end{array} \right\},$$

$$G(T^m(x), T^m(y), T^m(y)) \leq k \max\{G(x, T^m(y), T^m(y)), G(y, T^m(x), T^m(x)), G(y, T^m(y), T^m(y))\}, \quad (2.47)$$

or,

$$G(T^m(x), T^m(y), T^m(y)) \leq k \max\{G(x, x, T^m(y)), G(y, y, T^m(x)), G(y, y, T^m(y))\}, \quad (2.48)$$

for all $x, y, z \in X$, for some $m \in \mathbb{N}$, where $k \in [0, 1)$, then T has a unique fixed point (say u), and T^m is G -continuous at u .

Proof. The proof follows from Theorem 2.5, Corollary 2.6, and from an argument similar to that used in Corollary 2.2. \square

Theorem 2.8. *Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying one of the following conditions:*

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, T(y), T(y)), G(y, T(x), T(x))\} \quad (2.49)$$

or

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, x, T(y)), G(y, y, T(x))\}, \quad (2.50)$$

for all $x, y \in X$, where $k \in [0, 1)$, then T has a unique fixed point (say u), and T is G -continuous at u .

Proof. Since whenever the mapping satisfies condition (2.49), or (2.50), then it satisfies condition (2.45), or (2.46), respectively, in Theorem 2.5. Then the proof follows from Theorem 2.5. \square

Theorem 2.9. *Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$, be a mapping satisfying one of these conditions*

$$G(T(x), T(y), T(y)) \leq a\{G(x, T(y), T(y)) + G(y, T(x), T(x))\} \quad (2.51)$$

or

$$G(T(x), T(y), T(y)) \leq a\{G(x, x, T(y)) + G(y, y, T(x))\}, \quad (2.52)$$

for all $x, y \in X$, where $a \in [0, 1/2)$, then T has a unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.51), then we have

$$\begin{aligned} G(Tx, Ty, Ty) &\leq a\{G(y, Tx, Tx) + G(x, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq a\{G(x, Ty, Ty) + G(y, Tx, Tx)\}, \end{aligned} \quad (2.53)$$

for all $x, y \in X$.

Suppose that (X, G) is symmetric, then by definition of the metric (X, d_G) and (1.2), we get

$$d_G(Tx, Ty) \leq a\{d_G(x, Ty) + d_G(y, Tx)\} \quad \forall x, y \in X. \quad (2.54)$$

Since $0 \leq 2a < 1$, then the existence and uniqueness of the fixed point follow from a theorem in metric space (X, d_G) (see [13]).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (1.3), we have

$$d_G(Tx, Ty) \leq \frac{4a}{3}d_G(x, Ty) + \frac{4a}{3}d_G(y, Tx) \quad \forall x, y \in X. \quad (2.55)$$

So, the metric space (X, d_G) gives no information about this map since $8a/3$ need not be less than 1. But this can be proved by G -metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.51), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq a\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} = aG(x_{n-1}, x_{n+1}, x_{n+1}). \quad (2.56)$$

But

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + aG(x_n, x_{n+1}, x_{n+1}), \quad (2.57)$$

thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a}{1-a}G(x_{n-1}, x_n, x_n). \quad (2.58)$$

Let $k = a/(1-a)$, hence $0 \leq k < 1$ then continue in this procedure, we will get that $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$.

For all $n, m \in \mathbb{N}$; $n < m$, we have by rectangle inequality

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1})G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k}G(x_0, x_1, x_1). \end{aligned} \quad (2.59)$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$, and so, (x_n) is G-Cauchy sequence. By completeness of (X, G) , there exists $u \in X$ such that (x_n) is G-converge to u .

Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq a\{G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)\}. \quad (2.60)$$

Taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, then $G(u, T(u), T(u)) \leq aG(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v) = v$, then $G(u, v, v) \leq a\{G(u, v, v) + G(v, u, u)\}$, so

$$G(u, v, v) \leq \left(k = \frac{a}{1-a}\right)G(v, u, u) \quad (2.61)$$

again by the same argument, we can verify that $G(u, v, v) \leq k^2G(u, v, v)$, which implies that $u = v$.

To show that T is G-continuous at u , let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq a\{G(u, T(y_n), T(y_n)) + G(y_n, T(u), T(u))\}, \quad (2.62)$$

and so $G(u, T(y_n), T(y_n)) \leq (a/(1-a))G(y_n, T(u), T(u))$.

Taking the limit as $n \rightarrow \infty$, from which we see that $G(u, T(y_n), T(y_n)) \rightarrow 0$. By Proposition 1.7, we have $T(y_n) \rightarrow u = Tu$ which implies that T is G-continuous at u . \square

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