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Research Article

Iterative Approximation to Convex Feasibility Problems in Banach Space

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The convex feasibility problem (CFP) of finding a point in the nonempty intersection $\cap_{i=1}^N C_i$ is considered, where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set of a nonexpansive mapping $T_i : E \to E$, where E is a reflexive Banach space with a weakly sequentially continuous duality mapping. By using viscosity approximation methods for a finite family of nonexpansive mappings, it is shown that for any given contractive mapping $f : C \to C$, where C is a nonempty closed convex subset of E and for any given $x_0 \in C$ the iterative scheme $x_{n+1} = P[\alpha_{n+1} f(x_n) + (1-\alpha_{n+1})T_{n+1}x_n]$ is strongly convergent to a solution of (CFP), if and only if $\{\alpha_n\}$ and $\{x_n\}$ satisfy certain conditions, where $\alpha_n \in (0,1), T_n = T_{n(\text{mod}\, N)}$ and P is a sunny nonexpansive retraction of E onto C. The results presented in the paper extend and improve some recent results in Xu (2004), O'Hara et al. (2003), Song and Chen (2006), Bauschke (1996), Browder (1967), Halpern (1967), Jung (2005), Lions (1977), Moudafi (2000), Reich (1980), Wittmann (1992), Reich (1994).

1. Introduction

We are concerned with the following convex feasibility problem (CFP):

$$\text{finding an } x \in \bigcap_{i=1}^{N} C_i, \tag{1.1}$$

where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set of a nonexpansive mapping $T_i : E \to E$, $i = 1,2,\dots,N$. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [13–15], computer tomography [16], and radiation therapy treatment planning [17].

The aim of this paper is to study the CFP in the setting of Banach space. For that purpose, we first briefly state our iterative scheme and its history.

Let E be a Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_N$ be N nonexpansive mappings on E such that $C_i = F(T_i)$, the fixed point set of T_i. The iterative scheme that we are going to discuss is

$$x_{n+1} = P(\alpha_{n+1} f(x_n) + (1-\alpha_{n+1}) T_{n+1} x_n), \quad \forall n \geq 0, \tag{1.2}$$

where $x_0 \in E$ is any given initial data, $f(x) : C \to C$ is a given contractive mapping, $T_n = T_{n(\bmod N)}$, $\{\alpha_n\}$ is a sequence in $[0,1]$ and P is a sunny nonexpansive retraction of E onto C.

Next we consider some special cases of iterative scheme (1.2).

(1) If E is a Hilbert space, $f(x) \equiv u$ (a given point in C), $N = 1$ and T is a nonexpansive mapping on C, then the iterative scheme (1.2) is equivalent to the following iterative scheme:

$$x_{n+1} = \alpha_{n+1} u + (1-\alpha_{n+1}) T x_n, \quad \forall n \geq 0, \tag{1.3}$$

which was first introduced and studied by Halpern [6] in 1967. He proved that the iterative sequence (1.3) converges strongly to a fixed point of T, provided $\{\alpha_n\}$ satisfies certain conditions two of which are

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In 1992, Wittmann [11] proved that if $\{\alpha_n\}$ satisfies the conditions (C1), (C2), and the following condition:

(C4) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,

then the iterative sequence (1.3) converges strongly to a fixed point of T which improves and extends the corresponding results of Halpern [6], Lions [8].

In 1980 Reich [10] extended Halpern's result to all uniformly smooth Banach space and in 1994 he extended Wittmann's result to those uniformly smooth spaces with a weakly sequentially continuous duality mapping (see Reich [12, Theorem and Remark 1]). In 1997, Shioji and Takahashi [18] extended Wittmann's result to a wider class of Banach space.

(2) If E is a Hilbert space, C is a nonempty closed convex subset of E, $T_i : C \to C$ is a nonexpansive mapping, $i = 1, 2, \ldots, N$, and $f(x) = u$ (a given point in C), then (1.2) is equivalent to the following iterative sequence:

$$x_{n+1} = \alpha_{n+1} u + (1-\alpha_{n+1}) T_{n+1} x_n, \quad \forall n \geq 0, \tag{1.4}$$

(where $T_n = T_{n(\bmod N)}$) which was introduced and studied in Bauschke [4] in 1996. He proved that the iterative sequence (1.4) converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$, provided $\{\alpha_n\}$ satisfies conditions (C1), (C2), and the following condition:

(C5) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

(3) If E either is a uniformly smooth Banach space or a reflexive Banach space with a weakly sequentially continuous duality mapping and C a nonempty closed convex subset

of E. Assume that $T : C \to C$ is a nonexpansive mapping and $f : C \to C$ is a contractive mapping, then (1.2) is equivalent to the following sequence:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1-\alpha_{n+1}) T x_n, \quad \forall n \geq 0, \tag{1.5}$$

which was first introduced and studied by Moudafi [9] in the setting of Hilbert space. In 2004, Xu [1] extended and improved the corresponding results of Moudafi [9] to uniformly smooth Banach space and proved the following result.

Theorem 1.1 (Xu [1, Theorem 4.2]). *Let E be a uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping, let $x_0 \in C$ be any given point, let $\{\alpha_n\}$ be a real sequence in (0,1), and let $\{x_n\}$ be the iterative sequence defined by (1.5). If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$,

then $\{x_n\}$ converges strongly to a fixed point $p \in C$ of T which solves the following variational inequality:

$$\langle (I-f)p, J(p-u)\rangle \leq 0, \quad \forall u \in F(T). \tag{1.6}$$

Very recently, Song and Chen [3] extended Xu's result to the cases that T is a nonexpansive nonself-mapping and E is a reflexive Banach space with a weakly sequentially continuous duality mapping.

The purpose of this paper is by using viscosity approximation methods for a finite family of nonexpansive mappings to prove that for any given contractive mapping $f : C \to C$ and for any given $x_0 \in C$ the iterative scheme $\{x_n\}$ defined by (1.2) converges strongly to a solution of CFP, if and only if $\{\alpha_n\}$ and $\{x_n\}$ satisfy certain conditions, where $\alpha_n \in (0,1)$, $T_n = T_{n(\operatorname{mod} N)}$ and P is a sunny nonexpansive retraction of E onto C. The results presented in the paper extend and improve some recent results in Xu [1], O'Hara et al. [2], Song and Chen [3], Bauschke [4], Browder [5], Halpern [6], Jung [7], Lions [8], Moudafi [9], Reich [10], Wittmann [11], Reich [12].

2. Preliminaries

For the sake of convenience, we first recall some definitions, notations, and conclusions.

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, $F(T)$ is the set of fixed points of mapping T, $\langle \cdot, \cdot \rangle$ is the generalized duality pairing between E and E^*, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^*,\ \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\}, \quad x \in E. \tag{2.1}$$

When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \rightharpoonup^* x$) denotes strong (resp., weak and weak*) convergence of the sequence $\{x_n\}$ to x.

Defintion 2.1. (1) A mapping $f: C \to C$ is said to be a *Banach contraction on C with a contractive constant* $\beta \in (0,1)$ if $\|f(x) - f(y)\| \leq \beta\|x - y\|$ for all $x, y \in C$.

(2) Let $T: C \to C$ be a mapping. T is said to be *nonexpansive*, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

(3) Let $P: E \to C$ be a mapping. P is said to be

(a) *sunny*, if for each $x \in C$ and $t \in [0,1]$, we have

$$P(tx + (1-t)Px) = Px; \tag{2.3}$$

(b) *a retraction of E onto C*, if $Px = x$ for all $x \in C$;

(c) *a sunny nonexpansive retraction*, if P is sunny, nonexpansive retraction of E onto C;

(d) C is said to be a *sunny nonexpansive retract of E*, if there exists a sunny nonexpansive retraction of E onto C.

Defintion 2.2. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be a *smooth Banach space*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for each $x, y \in U$.

The following results give some characterizations of normalized duality mapping and sunny nonexpansive retractions on a smooth Banach space.

Lemma 2.3. (1) *A Banach space E is smooth if and only if the normalized duality mapping* $J: E \to 2^{E^*}$ *is single-valued. In this case, the normalized duality mapping J is strong-weak* continuous (see, e.g., [19]).*

(2) *Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. If* $P: E \to C$ *is a retraction and J is the normalized duality mapping on E, then the following conclusions are equivalent (see, [20–23]):*

(a) *P is sunny and nonexpansive;*

(b) $\|Px - Py\|^2 \leq \langle x - y, J(Px - Py)\rangle$ *for all* $x, y \in E$;

(c) $\langle x - Px, J(y - Px)\rangle \leq 0$ *for all* $x \in E$ *and* $y \in C$.

Remark 2.4. It should be pointed out that in the recent papers [24, 25] the authors deal with the construction of sunny nonexpansive retractions onto common fixed point sets of certain families of nonexpansive mappings in Banach spaces. Current information on (sunny) nonexpansive retracts in Banach spaces can be found in Kopecká and Reich [23].

Defintion 2.5 (Browder [5]). A Banach space is said to admit *a weakly sequentially continuous normalized duality mapping J*, if $J: E \to E^*$ is single-valued and weak-weak* sequentially continuous, that is, if $x_n \rightharpoonup x$ in E, then $J(x_n) \rightharpoonup^* J(x)$ in E^*.

The following results can be obtained from Definition 2.5.

LEMMA 2.6. *If E admits a weakly sequentially continuous normalized duality mapping, then*

(1) *E satisfies the Opial's condition, that is, whenever $x_n \rightharpoonup x$ in E and $y \neq x$, then* $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ *(see, Lim and Xu [26]).*

(2) *If $T : E \to E$ is a nonexpansive mapping, then the mapping $I - T$ is demiclosed, that is, for any sequence $\{x_n\}$ in E, if $x_n \rightharpoonup x$ and $(x_n - Tx_n) \to y$, then $(I - T)x = y$ (see, e.g., Goebel and Kirk [27]).*

Defintion 2.7. (1) Let C be a nonempty closed convex subset of a Banach space E. Then for each $x \in C$, the set $I_C(x)$ defined by

$$I_C(x) = \{y \in E : y = x + \lambda(z - x),\ z \in C,\ \lambda \geq 0\} \tag{2.5}$$

is called a *inward set*.

(2) A mapping $T : C \to E$ is said to satisfy *the weakly inward condition*, if $Tx \in \overline{I_C(x)}$ (the closure of $I_C(x)$) for each $x \in C$.

LEMMA 2.8. *Let E be a real smooth Banach space, let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E, and let P be a sunny nonexpansive retraction from E onto C. Let $T_i : E \to E$, $i = 1, 2, \dots, N$, be nonexpansive mappings satisfying the following conditions:*

(i) $\bigcap_{i=1}^{N} (F(T_i) \bigcap C) \neq \emptyset$;

(ii)

$$\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_N T_{N-1}, \dots, T_1) = F(S), \tag{2.6}$$

where

$$S = T_N T_{N-1}, \dots, T_1; \tag{2.7}$$

(iii) *$S : C \to E$ satisfies the weakly inward condition.*

Then $\bigcap_{i=1}^{N} (F(T_i) \bigcap C) = F(PS)$.

Proof. If $x \in \bigcap_{i=1}^{N} (F(T_i) \bigcap C)$, then $x = T_i x \in C$, $i = 1, 2, \dots, N$, and so $x = Sx \in C$. Since P is a sunny nonexpansive retraction from E onto C, we have $Px = PSx = x$. This implies that $x \in F(PS)$, and so $\bigcap_{i=1}^{N} (F(T_i) \bigcap C) \subset F(PS)$.

Conversely, if $x \in F(PS)$, then $x = PSx \in C$. Since P is a sunny nonexpansive retraction from E onto C, by Lemma 2.3(2)(c), we have

$$\langle Sx - x, J(y - x) \rangle \leq 0, \quad \forall y \in C. \tag{2.8}$$

By condition (iii), $Sx \in \overline{I_C(x)}$. Hence for each $n \geq 1$, there exist $z_n \in C$ and $\lambda_n \geq 0$ such that the sequence $y_n = x + \lambda_n(z_n - x) \to Sx$ $(n \to \infty)$. It follows from (2.8) and the positively homogeneous property of normalized duality mapping J that

$$\begin{aligned} 0 &\geq \lambda_n \langle Sx - x, J(z_n - x)\rangle \\ &= \langle Sx - x, J(\lambda_n(z_n - x))\rangle \\ &= \langle Sx - x, J(y_n - x)\rangle. \end{aligned} \tag{2.9}$$

Since E is smooth, it follows from Lemma 2.3(1) that the normalized duality mapping J is single-valued and strong-weak* continuous. Letting $n \to \infty$ in (2.9), we have

$$\begin{aligned} \|Sx - x\|^2 &= \langle Sx - x, J(Sx - x)\rangle \\ &= \lim_{n\to\infty} \langle Sx - x, J(y_n - x)\rangle \leq 0, \end{aligned} \tag{2.10}$$

that is, $x = Sx$. Since $x \in C$, we know that $x \in F(S) \bigcap C$. It follows from condition (ii) that $x \in \bigcap_{i=1}^N (F(x_i) \bigcap C)$. This shows that $F(PS) \subset \bigcap_{i=1}^N (F(x_i) \bigcap C)$.

The conclusion of Lemma 2.8 is proved. □

Lemma 2.9 [28]. *Let E be a real Banach space, and let $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following conclusions hold:*

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y); \\ \|x + y\|^2 &\geq \|x\|^2 + 2\langle y, j(x)\rangle, \quad \forall j(x) \in J(x). \end{aligned} \tag{2.11}$$

Lemma 2.10 (Liu [29]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0, \tag{2.12}$$

where n_0 is some nonnegative integer, $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^\infty \lambda_n = \infty$, $b_n = o(\lambda_n)$, and $\sum_{n=0}^\infty c_n < \infty$, then $a_n \to 0$ (as $n \to \infty$).

3. Main results

Let E be a real Banach space, let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E. Let $T_i : E \to E$, $i = 1, 2, \ldots, N$, be nonexpansive mappings and $f : C \to C$ a Banach contraction mapping with a contractive constant $0 < \beta < 1$. For given $t \in (0,1)$, define a mapping $S_t : C \to C$ by

$$S_t(x) = P(tf(x) + (1 - t)S(x)), \quad x \in C, \tag{3.1}$$

where P is the sunny nonexpansive retraction from E onto C and S is the mapping defined by (2.7). It is easy to see that $S_t : C \to C$ is a Banach contraction mapping. By Banach's contraction, principle yields a unique fixed point $z_t \in C$ of S_t, that is, z_t is the unique solution of the equation

$$z_t = P(tf(z_t) + (1 - t)S(z_t)), \quad t \in (0,1). \tag{3.2}$$

For the net $\{z_t\}$, we have the following result.

Theorem 3.1. *Let E be a real Banach space, let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E. Let $T_i : E \to E$, $i = 1,2,\dots,N$, be nonexpansive mappings, and let $f : C \to C$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$. Let $\{z_t : t \in (0,1)\}$ be the net defined by (3.2), where P is the sunny nonexpansive retraction of E onto C. If the following conditions are satisfied:*

(i) $\bigcap_{i=1}^{N}(F(T_i)\bigcap C) \neq \emptyset$;

(ii)

$$\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} F(T_1T_N\cdots T_3T_2) = \cdots = F(T_NT_{N-1},\dots,T_1) = F(S), \tag{3.3}$$

where $S = T_NT_{N-1},\dots,T_1$.

Then the following conclusions hold:

(1) $\langle z_t - f(z_t),\ j(z_t - u)\rangle \leq 0$, *for all* $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, *for all* $j(z_t - u) \in J(z_t - u)$;

(2) $\{z_t\}$ *is bounded.*

Proof. (1) For any $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, we have

$$(1-t)u + tf(z_t) = P((1-t)u + tf(z_t)). \tag{3.4}$$

Hence we have

$$\begin{aligned} ||z_t - [(1-t)u + tf(z_t)]|| &= ||P(tf(z_t) + (1-t)S(z_t)) - P((1-t)u + tf(z_t))|| \\ &\leq (1-t)||Sz_t - u|| \leq (1-t)||z_t - u||. \end{aligned} \tag{3.5}$$

By Lemma 2.9, we have

$$\begin{aligned} ||z_t - [(1-t)u + tf(z_t)]||^2 &= ||(1-t)(z_t - u) + t(z_t - f(z_t))||^2 \\ &\geq (1-t)^2||z_t - u||^2 + 2t\langle z_t - f(z_t), j((1-t)(z_t - u))\rangle \\ &= (1-t)^2||z_t - u||^2 + 2t(1-t)\langle z_t - f(z_t), j(z_t - u)\rangle. \end{aligned} \tag{3.6}$$

It follows from (3.5) that

$$\begin{aligned} &2t(1-t)\langle z_t - f(z_t), j(z_t - u)\rangle \\ &\quad \leq ||z_t - [(1-t)u + tf(z_t)]||^2 - (1-t)^2||z_t - u||^2 \leq 0. \end{aligned} \tag{3.7}$$

This shows that

$$\langle z_t - f(z_t), j(z_t - u)\rangle \leq 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big),\ \forall j(z_t - u) \in J(z_t - u). \tag{3.8}$$

(2) Since $f : C \to C$ is a Banach contraction mapping with a contractive constant $0 < \beta < 1$. Hence for any $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, we have

$$\langle f(z_t) - f(u), j(z_t - u)\rangle \leq \beta\|z_t - u\|^2. \tag{3.9}$$

Again since

$$\begin{aligned}\langle z_t - f(z_t), j(z_t - u)\rangle &= \langle z_t - u + u - f(u) + f(u) - f(z_t), j(z_t - u)\rangle \\ &= \|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle \\ &\quad + \langle f(u) - f(z_t), j(z_t - u)\rangle \\ &\geq \|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle \\ &\quad - \|f(u) - f(z_t)\|\|z_t - u\| \\ &\geq (1-\beta)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle.\end{aligned} \tag{3.10}$$

It follows from the conclusion (1) that

$$(1-\beta)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle \leq 0, \tag{3.11}$$

that is,

$$(1-\beta)\|z_t - u\|^2 \leq \langle u - f(u), j(u - z_t)\rangle \leq \|u - f(u)\| \cdot \|z_t - u\|. \tag{3.12}$$

Therefore we have

$$\|z_t - u\| \leq \frac{\|u - f(u)\|}{1-\beta}. \tag{3.13}$$

This shows that $\{z_t\}$ is bounded. □

Theorem 3.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E. Let $f : C \to C$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$, and let $T_i : E \to E$, $i = 1, 2, \ldots, N$, be nonexpansive mappings satisfying the following conditions:*

(i) $\bigcap_{i=1}^{N}(F(T_i)\bigcap C) \neq \emptyset$;

(ii)

$$\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} F(T_1T_N \cdots T_3T_2) = \cdots = F(T_NT_{N-1}, \ldots, T_1) = F(S), \tag{3.14}$$

where $S = T_NT_{N-1}, \ldots, T_1$;

(iii) *The mapping $S : C \to E$ satisfies the weakly inward condition.*

Let $\{z_t : t \in (0,1)\}$ be the net defined by (3.2), where P is the sunny nonexpansive retraction of E onto C. Then as $t \to 0$, $\{z_t\}$ converges strongly to some common fixed point

$p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$ *such that* p *is the unique solution of the following variational inequality:*

$$\langle (I-f)(p), J(p-u)\rangle \leq 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.15}$$

Proof. It follows from Theorem 3.1(2) that the net $\{z_t,\ t\in(0,1)\}$ is bounded and so $\{S(z_t),\ t\in(0,1)\}$ and $\{f(z_t),\ t\in(0,1)\}$ both are bounded. Hence from (3.2), we have

$$\begin{aligned} \|z_t - PS(z_t)\| &= \|P(tf(z_t)+(1-t)S(z_t)) - PSz_t\| \\ &\leq \|tf(z_t)+(1-t)S(z_t)-S(z_t)\| \\ &= t\|f(z_t)-S(z_t)\| \longrightarrow 0 \quad (\text{as } t \longrightarrow 0), \end{aligned} \tag{3.16}$$

and so we have

$$\lim_{t\to 0}\|z_t - PS(z_t)\| = 0. \tag{3.17}$$

Next we prove that $\{z_t : t\in(0,1)\}$ is relatively compact. Indeed, since E is reflexive and $\{z_t\}$ is bounded, for any subsequence $\{z_{t_n}\}\subset\{z_t\}$, there exists a subsequence of $\{z_{t_n}\}$ (for simplicity we still denote it by $\{z_{t_n}\}$) (where t_n is a sequence in (0,1)) such that $z_{t_n} \rightharpoonup p$ (as $t_n \to 0$). Since $PS : C \to C$ is nonexpansive, by virtue of (3.17) we have

$$\|z_{t_n} - PS(z_{t_n})\| \longrightarrow 0 \quad (\text{as } t_n \longrightarrow 0). \tag{3.18}$$

It follows from Lemma 2.6(2) that $I-PS$ has the demiclosed property, and so $p\in F(PS)$. Therefore it follows from Lemma 2.8 that

$$p \in F(PS) = \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.19}$$

Taking $u=p$ in (3.12), we have

$$\|z_{t_n} - p\|^2 \leq \frac{\langle p - f(p), J(p - z_{t_n})\rangle}{(1-\beta)}. \tag{3.20}$$

Since J is weakly sequentially continuous, we get that

$$\lim_{t_n\to 0}\|z_{t_n} - p\|^2 \leq \lim_{t_n\to 0}\frac{\langle p - f(p), J(p - z_{t_n})\rangle}{(1-\beta)} = 0, \tag{3.21}$$

that is, $z_{t_n} \to p$ (as $n\to\infty$).

This shows that $\{z_t\}$ is relatively compact.

Finally, we prove that the entire net $\{z_t,\ t\in(0,1)\}$ converges strongly to p.

Suppose the contrary that there exists another subsequence $\{z_{t_j}\}$ of $\{z_t\}$ such that $z_{t_j}\to q$ (as $t_j\to 0$). By the same method as given above, we can also prove that $q\in F(S)\bigcap C = \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$.

Next we prove that $p = q$ and p is the unique solution of the following variational inequality:

$$\langle (I-f)p, j(p-u)\rangle \le 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.22}$$

Indeed, for each $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, the sets $\{z_t - u\}$ and $\{z_t - f(z_t)\}$ both are bounded and the normalized duality mapping $J : E \to E^*$ is single-valued and weakly sequentially continuous. Hence it follows from $z_{t_j} \to q$ (as $t_j \to 0$) that

$$\begin{aligned}
&\big|\langle (I-f)(z_{t_j}), J(z_{t_j}-u)\rangle - \langle (I-f)(q), J(q-u)\rangle\big| \\
&\quad = \big|\langle (I-f)(z_{t_j}) - (I-f)(q), J(z_{t_j}-u)\rangle \\
&\qquad + \langle (I-f)(q), J(z_{t_j}-u) - J(q-u)\rangle\big| \\
&\quad \le \|(I-f)(z_{t_j}) - (I-f)(q)\| \cdot \|z_{t_j}-u\| \\
&\qquad + \big|\langle (I-f)(q), J(z_{t_j}-u) - J(q-u)\rangle\big| \longrightarrow 0 \quad (\text{as } t_j \longrightarrow 0).
\end{aligned} \tag{3.23}$$

By Theorem 3.1(1) we have

$$\langle (I-f)(q), J(q-u)\rangle = \lim_{t_j\to 0}\langle (I-f)(z_{t_j}), J(z_{t_j}-u)\rangle \le 0, \tag{3.24}$$

that is,

$$\langle (I-f)(q), J(q-u)\rangle \le 0. \tag{3.25}$$

Similarly we can also prove that

$$\langle (I-f)(p), J(p-u)\rangle \le 0. \tag{3.26}$$

Taking $u = p$ in (3.25) and $u = q$ in (3.26) and then adding up these two inequalities, we have

$$\langle (I-f)(p) - (I-f)(q), J(p-q)\rangle \le 0, \tag{3.27}$$

and so we have

$$\|p-q\|^2 \le \langle f(p) - f(q), J(p-q)\rangle \le \beta\|p-q\|^2. \tag{3.28}$$

This implies that $p = q$. The proof of Theorem 3.2 is completed. □

We are now in a position to prove the following result.

Theorem 3.3. *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E and P a sunny nonexpansive retraction from E onto C. Let $f : C \to C$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$, and let $T_i : E \to E$, $i = 1,2,\ldots,N$, be nonexpansive mappings satisfying the following conditions:*

(i) $\bigcap_{i=1}^{N}(F(T_i)\bigcap C) \neq \emptyset$;
(ii)

$$\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_N T_{N-1},\ldots,T_1) = F(S), \tag{3.29}$$

where $S = T_N T_{N-1},\ldots,T_1$;
(iii) *The mapping* $S : C \to E$ *satisfies the weakly inward condition.*
For any given $x_0 \in C$, *let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = P\big(\alpha_{n+1} f(x_n) + (1-\alpha_{n+1}) T_{n+1} x_n\big), \tag{3.30}$$

where $T_n = T_{n(\bmod N)}$. *Then the following hold.*

(1) $\{x_n\}$ *converges strongly to some* $p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$ *if and only if the following conditions are satisfied:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c) $\|x_n - PSx_n\| \to 0$.

(2) *If* $x_n \to p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, *then this* p *is the unique solution of the following variational inequality:*

$$\langle (I-f)(p), J(p-u)\rangle \leq 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.31}$$

Proof of conclusion (1) of Theorem 3.3 (sufficiency). (I) For the mapping $S : E \to E$ defined above, it is easy to see that S is a nonexpansive mapping. For given $f : C \to C$, $t \in (0,1)$, we define a contraction mapping $S_t : C \to C$ by

$$S_t x = P\big(tf(x) + (1-t)Sx\big), \quad x \in C, \tag{3.32}$$

where P is the sunny nonexpansive retraction from E onto C. Let $z_t \in C$ be the unique fixed point S_t, that is, it is the unique solution of the equation

$$z_t = P\big(tf(z_t) + (1-t)Sz_t\big). \tag{3.33}$$

By Theorem 3.2, $\{z_t,\ t \in (0,1)\}$ is bounded and as $t \to 0$, $\{z_t\}$ converges strongly to some $p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$ such that p is the unique solution of the following variational inequality:

$$\langle (I-f)(p), J(p-u)\rangle \leq 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.34}$$

(II) Now we prove that the sequence $\{x_n\}$ defined by (3.30) is bounded. In fact, for any $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$ and for any $n \geq 0$ we have

$$\begin{aligned}
||x_n - u|| &= ||P((1-\alpha_n)T_n x_{n-1} + \alpha_n f(x_{n-1})) - Pu|| \\
&\leq (1-\alpha_n)||T_n x_{n-1} - u|| + \alpha_n ||f(x_{n-1}) - u|| \\
&\leq (1-\alpha_n)||x_{n-1} - u|| + \alpha_n \{||f(x_{n-1}) - f(u)|| + ||f(u) - u||\} \\
&\leq (1-\alpha_n)||x_{n-1} - u|| + \alpha_n \{\beta ||x_{n-1} - u|| + ||f(u) - u||\} \\
&= (1-\alpha_n(1-\beta))||x_{n-1} - u|| + \alpha_n ||f(u) - u|| \\
&\leq \max\left\{||x_{n-1} - u||, \frac{||f(u) - u||}{1-\beta}\right\}.
\end{aligned} \tag{3.35}$$

By induction, we can prove that

$$||x_n - u|| \leq \max\left\{||x_0 - u||, \frac{||f(u) - u||}{1-\beta}\right\} \quad \forall n \geq 0. \tag{3.36}$$

This shows that $\{x_n\}$ is bounded. Let

$$M = \sup_{t\geq 0}\sup_{n\geq 0}\{||x_n - p||^2 + ||x_n - p|| + ||z_t - x_n|| + ||z_t - x_n||^2\} < \infty, \tag{3.37}$$

where p is the strong limit of the sequence $\{z_t\}$ defined by (3.33).

(III) Now we prove that

$$\limsup_{n\to\infty}\langle (I-f)(p), J(p - x_n)\rangle \leq 0. \tag{3.38}$$

Indeed, since E is reflexive and $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and

$$\limsup_{n\to\infty}\langle (I-f)(p), J(p - x_n)\rangle = \lim_{n_k\to\infty}\langle (I-f)(p), J(p - x_{n_k})\rangle. \tag{3.39}$$

By condition (c), we have

$$||x_{n_k} - PSx_{n_k}|| \longrightarrow 0 \quad (\text{as } n_k \longrightarrow \infty). \tag{3.40}$$

It follows from Lemmas 2.6(2) and 2.8 that

$$x^* \in F(PS) = \bigcap_{n=1}^{N}\left(F(T_i)\bigcap C\right). \tag{3.41}$$

Since the normalized duality J is weakly sequentially continuous, from (3.15) we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle (I-f)(p), J(p - x_n)\rangle \\
&\quad = \lim_{n_k\to\infty}\langle (I-f)(p), J(p - x_{n_k})\rangle = \langle (I-f)(p), J(p - x_*)\rangle \leq 0.
\end{aligned} \tag{3.42}$$

The conclusion (3.38) is proved.

(IV) *Letting* $\gamma_n = \max\{\langle (I-f)(p), J(p-x_n)\rangle, 0\} \geq 0$, *we prove that* $\gamma_n \to 0$ *(as* $n \to \infty$*).*

Indeed, it follows from (3.38) that for any given $\epsilon > 0$, there exists a positive integer n_1 such that

$$\langle (I-f)(p), J(p-x_n)\rangle < \epsilon \quad \forall n \geq n_1, \tag{3.43}$$

and so we have $0 \leq \gamma_n < \epsilon$, for all $n \geq n_1$. In view of the arbitrariness of $\epsilon > 0$, we know that $\gamma_n \to 0$.

(V) *Finally, we prove that* $x_n \to p$ where p is the limit of $\{z_t\}$ (as $t \to 0$) and $p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$. In fact, it follows from (3.30) and Lemma 2.9 that

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||P((1-\alpha_{n+1})T_{n+1}x_n + \alpha_{n+1}f(x_n)) - P(p)||^2 \\
&\leq ||(1-\alpha_{n+1})(T_{n+1}x_n - p) + \alpha_{n+1}(f(x_n)-p)||^2 \\
&\leq (1-\alpha_{n+1})^2||T_{n+1}x_n - p||^2 + 2\alpha_{n+1}\langle f(x_n)-p, J(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_{n+1})^2||x_n-p||^2 \\
&\quad + 2\alpha_{n+1}\langle f(x_n)-f(p)+f(p)-p, J(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_{n+1})^2||x_n-p||^2 + 2\alpha_{n+1}\beta||x_n-p|| \cdot ||x_{n+1}-p|| \\
&\quad + 2\alpha_{n+1}\langle f(p)-p, J(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_{n+1})^2||x_n-p||^2 + \alpha_{n+1}\beta\{||x_n-p||^2 + ||x_{n+1}-p||^2\} \\
&\quad + 2\alpha_{n+1}\langle f(p)-p, J(x_{n+1}-p)\rangle.
\end{aligned} \tag{3.44}$$

Since the normalized duality mapping J defined by (2.1) is odd, that is, $J(-x) = -J(x)$, $x \in E$, therefore we have

$$\langle f(p)-p, J(x_{n+1}-p)\rangle = \langle p-f(p), J(p-x_{n+1})\rangle \leq \gamma_{n+1}. \tag{3.45}$$

Substituting it into (3.44) and simplifying, we have

$$\begin{aligned}
(1-\beta\alpha_{n+1}) \cdot ||x_{n+1}-p||^2 &\leq [(1-\alpha_{n+1})^2 + \alpha_{n+1}\beta]||x_n-p||^2 \\
&\quad + 2\alpha_{n+1}\langle f(p)-p, J(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_{n+1}(2-\beta))||x_n-p||^2 \\
&\quad + \alpha_{n+1}^2||x_n-p||^2 + 2\alpha_{n+1}\gamma_{n+1} \\
&\leq (1-\alpha_{n+1}(2-\beta))||x_n-p||^2 + \alpha_{n+1}^2 M + 2\alpha_{n+1}\gamma_{n+1}.
\end{aligned} \tag{3.46}$$

Since $\alpha_n \to 0$, therefore there exists a positive integer n_2 such that

$$1-\beta\alpha_{n+1} > \frac{1}{2}, \quad \forall n \geq n_2. \tag{3.47}$$

It follows from (3.46) that

$$\begin{aligned}||x_{n+1}-z||^2 \leq \frac{1-\alpha_{n+1}(2-\beta)}{1-\beta\alpha_{n+1}}||x_n-p||^2 \\ +2\alpha_{n+1}\{\alpha_{n+1}M+2\gamma_{n+1}\} \quad \forall n \geq n_2.\end{aligned} \tag{3.48}$$

Again since

$$\frac{1-\alpha_{n+1}(2-\beta)}{1-\beta\alpha_{n+1}} = 1-\frac{2\alpha_{n+1}(1-\beta)}{1-\beta\alpha_{n+1}} \leq 1-2\alpha_{n+1}(1-\beta), \tag{3.49}$$

from (3.48) we have

$$\begin{aligned}||x_{n+1}-z||^2 \leq \{1-2\alpha_{n+1}(1-\beta)\}||x_n-p||^2 \\ +2\alpha_{n+1}\{\alpha_{n+1}M+2\gamma_{n+1}\} \quad \forall n \geq n_2.\end{aligned} \tag{3.50}$$

Take $a_n = \|x_n - p\|^2$, $\lambda_n = 2\alpha_{n+1}(1-\beta)$, $b_n = 2\alpha_{n+1}\{\alpha_{n+1}M + 2\gamma_{n+1}\}$, and $c_n = 0$ for all $n \geq n_2$ in Lemma 2.10. By the assumptions, it is easy to see that $\sum_{n=0}^{\infty}\lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n = 0$, hence the conditions in Lemma 2.10 are satisfied, and so we have

$$\lim_{n\to\infty}||x_n-p|| = 0, \quad \text{that is, } x_n \longrightarrow p \in \bigcap_{i=}^{N}\Big(F(T_i)\bigcap C\Big). \tag{3.51}$$

The sufficiency of conclusion (1) of Theorem 3.3 is proved.

(Necessity). Suppose that the sequence $\{x_n\}$ defined by (3.30) converges strongly to a fixed point $p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C) = F(S)\bigcap C$. Therefore we have that

$$\begin{aligned}||PSx_n-x_n|| &= ||PSx_n-Px_n|| \leq ||Sx_n-x_n|| \\ &\leq ||Sx_n-p||+||x_n-p|| \\ &\leq 2||x_n-p|| \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{3.52}$$

The necessity of condition (c) is proved.

Now we prove the necessity of condition (a). Take $f \equiv u$, $u \in C$ with $u \neq p$ and T_i is nonexpansive mapping from C to C for all $i = 1,2,\dots,N$. Then the sequence $\{x_n\}$ defined by (3.30) is equivalent to the following iterative sequence:

$$\begin{aligned}x_0 &\in C, \\ x_{n+1} &= \alpha_{n+1}u+(1-\alpha_{n+1})T_{n+1}x_n,\end{aligned} \tag{3.53}$$

where $T_n = T_{n(\bmod N)}$. Since each $T_i : C \to C$, $i = 1,2,\dots,N$, is nonexpansive, we get

$$||T_{n+1}x_n - p|| \le ||x_n - p|| \longrightarrow 0, \quad \text{that is, } T_{n+1}x_n \longrightarrow p \text{ (as } n \longrightarrow \infty). \tag{3.54}$$

Again from (3.53) we have that

$$\begin{aligned} \alpha_{n+1}||u - T_{n+1}x_n|| &= ||x_{n+1} - T_{n+1}x_n|| \\ &\le ||x_{n+1} - p|| + ||T_{n+1}x_n - p|| \\ &\le ||x_{n+1} - p|| + ||x_n - p|| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned} \tag{3.55}$$

Therefore we have

$$\limsup_{n\to\infty} \alpha_{n+1}||u - T_{n+1}x_n|| = \limsup_{n\to\infty} \alpha_{n+1}\|u - p\| = 0. \tag{3.56}$$

By the assumption that $u \ne p$, and so we have

$$\limsup_{n\to\infty} \alpha_{n+1} = 0, \tag{3.57}$$

that is,

$$\lim_{n\to\infty} \alpha_n = 0. \tag{3.58}$$

The necessity of condition (a) is proved.

Take $f = 0$, $C = \{x \in E : \|x\| \le 1\}$ (closed unit ball in E), and $T_i = (-I) : C \to C$, for all $i = 1,2,\dots,N$, in (3.30), where I is the identity mapping. Since each T_i, $i = 1,2,\dots,N$, is nonexpansive and 0 is the unique common fixed point of $T_1, T_2, \dots, T_N$ in C, hence we have

$$x_{n+1} = (-1)(1 - \alpha_{n+1})x_n = (-1)^2(1 - \alpha_{n+1})(1 - \alpha_n)x_{n-1} = \cdots = (-1)^{n+1}\prod_{i=1}^{n+1}(1 - \alpha_i)x_0. \tag{3.59}$$

If $x_n \to 0 \in \bigcap_{i=1}^{N} F_{ix}(T_i)$, we have

$$0 = \lim_{n\to\infty} ||x_{n+1} - 0|| = \lim_{n\to\infty}\prod_{i=1}^{n+1}(1 - \alpha_i)||x_0 - 0||. \tag{3.60}$$

This implies that

$$\prod_{i=1}^{\infty}(1 - \alpha_i) = 0, \quad \text{that is, } \sum_{i=1}^{\infty}\alpha_i = \infty. \tag{3.61}$$

The necessity of condition (b) is proved.

Summing up the about argument, the conclusion (1) of Theorem 3.3 is proved.

The conclusion (2) of Theorem 3.3 can be obtained from Theorem 3.2 immediately.

The proof of Theorem 3.3 is completed. □

4. Applications to some recent theorems

As applications of Theorem 3.3 we can obtain the following results.

Theorem 4.1. *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E and P a sunny nonexpansive retraction from E onto C. Let $f : C \to C$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$, and let $T_i : E \to E$, $i = 1,2,\dots,N$, be nonexpansive mappings satisfying the conditions (i), (ii) and (iii) in Theorem 3.1. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (3.30). If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$,

then the sequence $\{x_n\}$ converges strongly to a point $p \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$ which is the unique solution of the following variational inequality:

$$\langle (I - f)(p), J(p - u)\rangle \le 0, \quad \forall u \in \bigcap_{i=1}^{N}\Big(F(T_i)\bigcap C\Big). \tag{4.1}$$

Proof. It suffices to prove that under the conditions of Theorem 4.1 the condition $\|x_n - PSx_n\| \to 0$ (as $n \to \infty$) in Theorem 3.3 is satisfied.

In fact, for given contraction mapping $f : C \to C$ with a contractive constant $\beta \in (0,1)$ and for given point $u \in \bigcap_{i=1}^{N}(F(T_i)\bigcap C)$, by the same method as given in the proof of (3.36), we can prove that

$$||x_n - u|| \le \max\left\{||x_0 - u||, \frac{||f(u) - u||}{1-\beta}\right\} \quad \forall n \ge 0. \tag{4.2}$$

This implies that $\{x_n\}$ is bounded, and so $\{Sx_n\}$ and $\{f(x_n)\}$ both are bounded. Let

$$M = \sup_{n\ge 0}\{||f(x_n) - Sx_n|| + ||f(x_n) - S(x_{n+1})||\}. \tag{4.3}$$

It follows from (3.30) that

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||P(\alpha_{n+1} f(x_n) + (1-\alpha_{n+1})Sx_n) - P(\alpha_n f(x_{n-1}) + (1-\alpha_n)Sx_{n-1})|| \\
&\le ||\alpha_{n+1} f(x_n) + (1-\alpha_{n+1})Sx_n - (\alpha_n f(x_{n-1}) + (1-\alpha_n)Sx_{n-1})|| \\
&= ||(1-\alpha_{n+1})(Sx_n - Sx_{n-1}) + (\alpha_{n+1} - \alpha_n)(f(x_{n-1}) - Sx_{n-1}) \\
&\quad + \alpha_{n+1}(f(x_n) - f(x_{n-1}))|| \\
&\le (1-\alpha_{n+1})||x_n - x_{n-1}|| + |\alpha_{n+1} - \alpha_n|\,||f(x_{n-1}) - Sx_{n-1}|| \\
&\quad + \alpha_{n+1}\beta||x_n - x_{n-1}|| \\
&\le (1 - \alpha_{n+1}(1-\beta))||x_n - x_{n-1}|| + M|\alpha_{n+1} - \alpha_n|.
\end{aligned} \tag{4.4}$$

(I) If the condition $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ is satisfied, then take $a_n=\|x_n-x_{n-1}\|$, $\lambda_n=\alpha_{n+1}(1-\beta)$, $b_n=0$, and $c_n=M|\alpha_{n+1}-\alpha_n|$ in Lemma 2.10. It is easy to see that all conditions in Lemma 2.10 are satisfied. Therefore we have

$$||x_{n+1}-x_n||\longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{4.5}$$

(II) If the condition $\lim_{n\to\infty}\alpha_n/\alpha_{n+1}=1$ is satisfied, then take $a_n=\|x_n-x_{n-1}\|$, $\lambda_n=\alpha_{n+1}(1-\beta)$, $c_n=0$ and

$$b_n=M\left|\alpha_{n+1}-\alpha_n\right|=M\alpha_{n+1}\left|\frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}}\right|=M\alpha_{n+1}\left|1-\frac{\alpha_n}{\alpha_{n+1}}\right|=0(\lambda_n). \tag{4.6}$$

It is easy to see that all conditions in Lemma 2.10 are satisfied. Therefore (4.5) still holds.

From (3.30), we have

$$\begin{aligned}||x_n-PSx_n||&=||P(\alpha_n f(x_{n-1})+(1-\alpha_n)Sx_{n-1})-PSx_n||\\&\le||\alpha_n f(x_{n-1})+(1-\alpha_n)Sx_{n-1}-Sx_n||\\&=||\alpha_n[f(x_{n-1})-Sx_n]+(1-\alpha_n)(Sx_{n-1}-Sx_n)||\\&\le\alpha_n||f(x_{n-1})-Sx_n||+(1-\alpha_n)||Sx_{n-1}-Sx_n||\\&\le\alpha_n M+(1-\alpha_n)||x_{n-1}-x_n||.\end{aligned} \tag{4.7}$$

By virtue of (4.5) and condition (i), we obtain

$$\lim_{n\to\infty}||x_n-PSx_n||\longrightarrow 0. \tag{4.8}$$

This shows that the condition (iii) in Theorem 3.3 is satisfied. Hence the conclusion of Theorem 4.1 can be obtained from Theorem 3.3 immediately. □

THEOREM 4.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from E to E^*. Let C be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E and P a sunny nonexpansive retraction from E onto C. Let $f:C\to C$ be a given Banach contraction mapping with a contractive constant $0<\beta<1$, and let $T:C\to E$ be a nonexpansive mapping with $F(T)\neq\emptyset$ and satisfy the weakly inward condition.*

For any given $x_0\in C$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1}=P(\alpha_{n+1}f(x_n)+(1-\alpha_{n+1})Tx_n). \tag{4.9}$$

Then the following hold.

(1) $\{x_n\}$ converges strongly to a point $p\in F(T)$ if and only if the following conditions are satisfied:

(a) $\lim_{n\to\infty}\alpha_n=0$;

(b) $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(c) $\|x_n-PTx_n\|\to 0$.

(2) If $x_n\to p\in F(T)$, then p is the unique solution of the following variational inequality:

$$\langle(I-f)(p),J(p-u)\rangle\le 0,\quad \forall u\in F(T). \tag{4.10}$$

Proof. The conclusion of Theorem 4.2 can be obtained from Theorem 3.3 immediately. □

Remark 4.3. Theorem 4.2 improves and extends the main results in Xu [1], O'Hara et al. [2], Song and Chen [3], Jung [7], Wittmann [11], Bauschke [4], Moudafi [9], Lions [8], Halpern [6], Reich [10, 12], and Browder [5].

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