BROWDER'S TYPE STRONG CONVERGENCE THEOREMS FOR INFINITE FAMILIES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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We prove Browder's type strong convergence theorems for infinite families of nonexpansive mappings. One of our main results is the following: let C be a bounded closed convex subset of a uniformly smooth Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0,1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n/t_n^{\ell} = 0$ for $\ell \in \mathbb{N}$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1-\alpha_n)((1-\sum_{k=1}^n t_n^k)T_1 u_n + \sum_{k=1}^n t_n^k T_{k+1} u_n) + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^\infty F(T_n)$.

1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty in the case that E is uniformly smooth and C is bounded; see Baillon [1]. When E has the Opial property and C is weakly compact, $F(T)$ is also nonempty; see [11, 13]. See also [4, 5, 10] and others. Fix $u \in C$. Then for each $\alpha \in (0,1)$, there exists a unique point x_α in C satisfying $x_\alpha = (1-\alpha)Tx_\alpha + \alpha u$ because the mapping $x \mapsto (1-\alpha)Tx + \alpha u$ is contractive; see [2]. In 1967 Browder [6] proved the following strong convergence theorem.

Theorem 1.1 (Browder [6]). *Let C be a bounded closed convex subset of a Hilbert space E and let T be a nonexpansive mapping on C. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1-\alpha_n)Tu_n + \alpha_n u \tag{1.1}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(T)$ nearest to u.

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Reich extended this theorem to uniformly smooth Banach spaces in [17]. Using the notion of Bochner integral and (invariant) mean, Shioji and Takahashi in [18] proved Browder's type strong convergence theorems for families of nonexpansive mappings

Very recently, the author proved the following Browder's type strong convergence theorem for one-parameter nonexpansive semigroups. This is a generalization of the results in [19, 25]. We remark that we do not use the notion of Bochner integral.

Theorem 1.2 [24]. *Let C be a weakly compact convex subset of a Banach space E. Assume that either of the following holds:*

(i) *E is uniformly convex with uniformly Gâteaux differentiable norm;*

(ii) *E is uniformly smooth; or*

(iii) *E is a smooth Banach space with the Opial property and the duality mapping J of E is weakly sequentially continuous at zero.*

Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $0 < \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n) T(\tau + t_n) u_n + \alpha_n u \tag{1.2}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{t \geq 0} F(T(t))$.

Also, very recently, the author proved Krasnoselskii and Mann's type convergence theorems for infinite families of nonexpansive mappings in [21]. See also [20]. In this paper, using the idea in [21], we prove Browder's type strong convergence theorems for infinite families of nonexpansive mappings without assuming the strict convexity of the Banach space. We remark that if we assume the strict convexity, its proof is very easy because the set of common fixed points of countable families of nonexpansive mappings is the set of fixed points of some single nonexpansive mapping; see Bruck [8]. We also remark that we do not use the notion of (invariant) mean.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of all positive integers, all integers, all rational numbers, and all real numbers, respectively.

Let $\{x_n\}$ be a sequence in a topological space X. By the axiom of choice, there exist a directed set $(D, \leq)$ and a *universal subnet* $\{x_{f(\nu)} : \nu \in D\}$ of $\{x_n\}$, that is,

(i) f is a mapping from D into $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $\nu_0 \in D$ such that $\nu \geq \nu_0$ implies $f(\nu) \geq n$;

(ii) for each subset A of X, there exists $\nu_0 \in D$ such that either $\{x_{f(\nu)} : \nu \geq \nu_0\} \subset A$ or $\{x_{f(\nu)} : \nu \geq \nu_0\} \subset X \setminus A$ holds.

In this paper, we often use $\{x_\nu : \nu \in D\}$ instead of $\{x_{f(\nu)} : \nu \in D\}$, for short. We know that if a net $\{x_\nu\}$ is universal and g is a mapping from X into an arbitrary set Y, then $\{g(x_\nu)\}$ is also universal. We also know that if X is compact, then a universal net $\{x_\nu\}$ always converges. See [12] for details.

Let E be a real Banach space. We denote by E^* the dual of E. E is called *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. E is said to be *smooth* or said to have a *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in E$ with $\|y\| = 1$, the limit is attained uniformly in $x \in E$ with $\|x\| = 1$. E is said to be *uniformly smooth* or said to have a *uniformly Fréchet differentiable norm* if the limit is attained uniformly in $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have the *Opial property* [14] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0,

$$\liminf_{n \to \infty} \|x_n - x_0\| < \liminf_{n \to \infty} \|x_n - x\| \tag{2.2}$$

holds for all $x \in E$ with $x \neq x_0$. We remark that we may replace "liminf" by "limsup." That is, E has the Opial property if and only if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0,

$$\limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - x\| \tag{2.3}$$

holds for all $x \in E$ with $x \neq x_0$.

Let E be a smooth Banach space. The *duality mapping* J from E into E^* is defined by

$$\langle x, J(x) \rangle = \|x\|^2 = \|J(x)\|^2 \tag{2.4}$$

for all $x \in E$. J is said to be *weakly sequentially continuous at zero* if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

A convex subset C of a Banach space E is said to have *normal structure* [3] if for every bounded convex subset K of C which contains more than one point, there exists $z \in K$ such that

$$\sup_{x \in K} \|x - z\| < \sup_{x, y \in K} \|x - y\|. \tag{2.5}$$

We know that compact convex subsets of any Banach spaces and closed convex subsets of uniformly convex Banach spaces have normal structure. Turett [27] proved that uniformly smooth Banach spaces have normal structure. Also, Gossez and Lami Dozo [11] proved that every weakly compact convex subset of a Banach space with the Opial property has normal structure. We recall that a closed convex subset C of a Banach space E is said to have the *fixed point property for nonexpansive mappings* (*FPP*, for short) if for every bounded closed convex subset K of C, every nonexpansive mapping on K has a fixed point. So, by Kirk's fixed point theorem [13], every weakly compact convex subset with normal structure has FPP.

Let C and K be subsets of a Banach space E. A mapping P from C into K is called *sunny* [7] if

$$P(Px + t(x - Px)) = Px \tag{2.6}$$

for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$. The following is proved in [15].

Lemma 2.1 (Reich [15]). *Let E be a smooth Banach space and let C be a convex subset of E. Let K be a subset of C and let P be a retraction from C onto K. Then the following are equivalent:*

(i) $\langle x - Px, J(Px - y)\rangle \geq 0$ *for all* $x \in C$ *and* $y \in K$;

(ii) *P is both sunny and nonexpansive.*

Hence, there is at most one sunny nonexpansive retraction from C onto K.

The following lemma is proved in [24]. However, it is essentially proved in [16]. See also [26].

Lemma 2.2 (Reich [16]). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_\alpha : \alpha \in D\}$ be a net in E and let $z \in C$. Suppose that the limits of $\{\|x_\alpha - y\|\}$ exist for all $y \in C$. Then the following are equivalent:*

(i) $\lim_{\alpha \in D} \|x_\alpha - z\| = \min_{y \in C} \lim_{\alpha \in D} \|x_\alpha - y\|$;

(ii) $\limsup_{\alpha \in D} \langle y - z, J(x_\alpha - z)\rangle \leq 0$ *for all* $y \in C$;

(iii) $\liminf_{\alpha \in D} \langle y - z, J(x_\alpha - z)\rangle \leq 0$ *for all* $y \in C$.

The following lemma is well known.

Lemma 2.3. *Let $\{u_n\}$ be a sequence in a Banach space E and let z belong to E. Assume that every subsequence $\{u_{n_i}\}$ of $\{u_n\}$ has a subsequence converging to z. Then $\{u_n\}$ itself converges to z.*

From Lemma 2.3, we obtain the following.

Lemma 2.4. *Let $\{u_n\}$ be a sequence in a Banach space E. Assume that $\{u_n\}$ has at most one cluster point, and every subsequence of $\{u_n\}$ has a cluster point. Then $\{u_n\}$ converges.*

Proof. Since $\{u_n\}$ is a subsequence of $\{u_n\}$, $\{u_n\}$ has a cluster point $z \in E$. Let $\{u_{n_i}\}$ be an arbitrary subsequence of $\{u_n\}$. Then by assumption $\{u_{n_i}\}$ has a cluster point $w \in E$. Since w is also a cluster point of $\{u_n\}$, we have $w = z$. Hence, $\{u_{n_i}\}$ has a cluster point $z \in E$. That is, there exists a subsequence of $\{u_{n_i}\}$ converging to z. So, by Lemma 2.3, $\{u_n\}$ converges to z. This completes the proof. □

3. Fixed point theorem

The following theorem is one of the most famous fixed point theorems for families of nonexpansive mappings.

Theorem 3.1 (Bruck [9]). *Suppose a closed convex subset C of a Banach space E has the fixed point property for nonexpansive mappings, and C is either weakly compact, or bounded and separable. Then for any commuting family S of nonexpansive mappings on C, the set of common fixed points of S is a nonempty nonexpansive retract of C.*

Using Theorem 3.1, we prove the following fixed point theorem.

Theorem 3.2. *Let C be a closed convex subset of a Banach space E. Let A be a weakly compact convex subset of C. Assume that A has the fixed point property for nonexpansive mappings. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C such that*

$$T_1(A) \subset A, \qquad T_{\ell+1}\left(A \cap \left(\bigcap_{k=1}^{\ell} F(T_k)\right)\right) \subset A \tag{3.1}$$

for all $\ell \in \mathbb{N}$. Then there exists a common fixed point $z_0 \in A$ of $\{T_n : n \in \mathbb{N}\}$.

Proof. We put $B_\ell := A \cap (\bigcap_{k=1}^{\ell} F(T_k))$ for $\ell \in \mathbb{N}$. We first show B_ℓ is nonempty and there exists a nonexpansive retraction P_ℓ from A onto B_ℓ for all $\ell \in \mathbb{N}$. From the assumption of $T_1(A) \subset A$, there exists a fixed point $z_1 \in A$ of T_1, that is, $B_1 \neq \varnothing$. By Theorem 3.1, there exists a nonexpansive retraction P_1 from A onto B_1. We assume B_ℓ is nonempty and there exists a nonexpansive retraction P_ℓ from A onto B_ℓ for some $\ell \in \mathbb{N}$. From the assumption of $T_{\ell+1}(B_\ell) \subset A$, we have that $T_{\ell+1} \circ P_\ell$ is a nonexpansive mapping on A. We note that $B_{\ell+1} = F(T_{\ell+1} \circ P_\ell)$. Indeed, $B_{\ell+1} \subset F(T_{\ell+1} \circ P_\ell)$ is obvious. Conversely, we assume $z_2 \in A$ satisfies $T_{\ell+1} \circ P_\ell z_2 = z_2$. For $k \in \mathbb{N}$ with $k \leq \ell$, we have

$$T_k z_2 = T_k \circ T_{\ell+1} \circ P_\ell z_2 = T_{\ell+1} \circ T_k \circ P_\ell z_2 = T_{\ell+1} \circ P_\ell z_2 = z_2, \tag{3.2}$$

that is, $z_2 \in B_\ell$ and hence $P_\ell z_2 = z_2$. Thus, we also have

$$T_{\ell+1} z_2 = T_{\ell+1} \circ P_\ell z_2 = z_2. \tag{3.3}$$

Therefore $z_2 \in B_{\ell+1}$ and hence $B_{\ell+1} \supset F(T_{\ell+1} \circ P_\ell)$. We have shown $B_{\ell+1} = F(T_{\ell+1} \circ P_\ell)$. Since A has the fixed point property, we have

$$B_{\ell+1} = F(T_{\ell+1} \circ P_\ell) \neq \varnothing. \tag{3.4}$$

By Theorem 3.1 again, there exists a nonexpansive retraction $P_{\ell+1}$ from A onto $B_{\ell+1}$. So, by induction, we have shown that B_ℓ is nonempty and there exists a nonexpansive retraction P_ℓ from A onto B_ℓ for all $\ell \in \mathbb{N}$. Define a sequence $\{Q_n : n \in \mathbb{N}\}$ of nonexpansive mappings on A by

$$Q_n := P_n \circ P_{n-1} \circ \cdots \circ P_2 \circ P_1 \tag{3.5}$$

for $n \in \mathbb{N}$. Since $P_m x = P_n \circ P_m x$ for $x \in A$, $m, n \in \mathbb{N}$ with $m \geq n$, we have

$$Q_m \circ Q_n = P_{\max\{m,n\}} \circ P_{\max\{m,n\}-1} \circ \cdots \circ P_2 \circ P_1 \tag{3.6}$$

for all $m, n \in \mathbb{N}$ and hence $Q_m \circ Q_n = Q_n \circ Q_m$ for all $m, n \in \mathbb{N}$. So, by Theorem 3.1, there exists a common fixed point $z_0 \in A$ of $\{Q_n : n \in \mathbb{N}\}$. Let us prove that z_0 is also a common fixed point of $\{T_n : n \in \mathbb{N}\}$. Since

$$P_1 z_0 = Q_1 z_0 = z_0, \tag{3.7}$$

we have $z_0 \in B_1$, that is, $T_1 z_0 = z_0$. We assume

$$T_1 z_0 = T_2 z_0 = \cdots = T_\ell z_0 = z_0 \tag{3.8}$$

for some $\ell \in \mathbb{N}$. Then

$$\begin{aligned} z_0 &= Q_{\ell+1} z_0 = P_{\ell+1} \circ P_\ell \circ \cdots \circ P_2 \circ P_1 z_0 \\ &= P_{\ell+1} \circ P_\ell \circ \cdots \circ P_2 z_0 = \cdots = P_{\ell+1} \circ P_\ell z_0 = P_{\ell+1} z_0 \end{aligned} \tag{3.9}$$

and hence $z_0 \in B_{\ell+1}$, that is, $T_{\ell+1} z_0 = z_0$. So, by induction, z_0 is a common fixed point of $\{T_n : n \in \mathbb{N}\}$. This completes the proof. □

4. Lemmas

In this section, we prove some lemmas which are used in the proofs of our main results.

Lemma 4.1. *Let C be a closed convex subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C with a common fixed point. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0,1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n / t_n^\ell = 0$ for $\ell \in \mathbb{N}$. Let $\{I_n\}$ be a sequence of nonempty subsets of $\mathbb{N}$ such that $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty I_n = \mathbb{N}$. For $I \subset \mathbb{N}$ and $t \in (0,1/2)$ with $I \neq \varnothing$, define nonexpansive mappings $S(I,t)$ on C by*

$$S(I,t)x := \left(\left(1 - \sum_{k \in I} t^k\right) T_1 x + \sum_{k \in I} t^k T_{k+1} x\right) \tag{4.1}$$

for $x \in C$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n) S(I_n, t_n) u_n + \alpha_n u \tag{4.2}$$

for $n \in \mathbb{N}$. Let $\{u_{n_\beta} : \beta \in D\}$ be a subnet of $\{u_n\}$. Then the following hold.

(i) $\limsup_\beta \|u_{n_\beta} - T_1 x\| \leq \limsup_\beta \|u_{n_\beta} - x\|$ *for $x \in C$.*

(ii) *If $x \in C$ satisfies $T_1 x = x$, then* $\limsup_\beta \|u_{n_\beta} - T_2 x\| \leq \limsup_\beta \|u_{n_\beta} - x\|$.

(iii) *If $x \in C$ satisfies $T_1 x = T_2 x = \cdots = T_{\ell-1} x = x$ for some $\ell \in \mathbb{N}$ with $\ell \geq 3$, then* $\limsup_\beta \|u_{n_\beta} - T_\ell x\| \leq \limsup_\beta \|u_{n_\beta} - x\|$.

Proof. Let v be a common fixed point of $\{T_n : n \in \mathbb{N}\}$. It is obvious that $S(I,t)v = v$ for all $I \subset \mathbb{N}$ and $t \in (0,1/2)$ with $I \neq \varnothing$. For $x \in C$ and $k \in \mathbb{N}$, we have

$$\|T_k x\| \leq \|T_k x - v\| + \|v\| = \|T_k x - T_k v\| + \|v\| \leq \|x - v\| + \|v\|. \tag{4.3}$$

Hence, $\{T_k x : k \in \mathbb{N}\}$ is bounded for every $x \in C$. Therefore $S(I,t)$ is well defined for every $I \subset \mathbb{N}$ and $t \in (0,1/2)$ with $I \neq \varnothing$. It is obvious that $S(I,t)$ is a nonexpansive mapping on C for every I and t. Since

$$\begin{aligned} \|u_n - v\| &= \|(1 - \alpha_n) S(I_n, t_n) u_n + \alpha_n u - v\| \\ &\leq (1 - \alpha_n) \|S(I_n, t_n) u_n - v\| + \alpha_n \|u - v\| \\ &\leq (1 - \alpha_n) \|u_n - v\| + \alpha_n \|u - v\|, \end{aligned} \tag{4.4}$$

we have $\|u_n - v\| \le \|u - v\|$ for $n \in \mathbb{N}$. Therefore $\{u_n\}$ is bounded. Since

$$\|T_k u_n\| \le \|T_k u_n - v\| + \|v\| \le \|u_n - v\| + \|v\| \le \|u_n\| + 2\|v\| \tag{4.5}$$

for all $n, k \in \mathbb{N}$, $\{T_k u_n : n, k \in \mathbb{N}\}$ is also bounded. We fix $x \in C$ and we put

$$M := \max\left\{\|u\|, \|v\|, \sup_{n\in\mathbb{N}}\|u_n\|, \sup_{n,k\in\mathbb{N}}\|T_k u_n\|, \|x\|, \sup_{k\in\mathbb{N}}\|T_k x\|\right\} < \infty. \tag{4.6}$$

It is obvious that $\|S(I,t)u_n\| \le M$ and $\|S(I,t)x\| \le M$ for all $n \in \mathbb{N}$, $I \subset \mathbb{N}$ and $t \in (0,1/2)$ with $I \neq \varnothing$. From the assumption, we have

$$S(I_n, t_n)u_n - u_n = \alpha_n\big(S(I_n, t_n)u_n - u\big) \tag{4.7}$$

for $n \in \mathbb{N}$. We have

$$\begin{aligned}
\|u_{n_\beta} - T_1 x\| &\le \|u_{n_\beta} - S(I_{n_\beta}, t_{n_\beta})u_{n_\beta}\| + \|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - S(I_{n_\beta}, t_{n_\beta})x\| + \|S(I_{n_\beta}, t_{n_\beta})x - T_1 x\| \\
&\le \alpha_{n_\beta}\|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - u\| + \|u_{n_\beta} - x\| + \left\| -\sum_{k\in I_{n_\beta}} t_{n_\beta}^k T_1 x + \sum_{k\in I_{n_\beta}} t_{n_\beta}^k T_{k+1}x\right\| \\
&\le 2M\alpha_{n_\beta} + \|u_{n_\beta} - x\| + 2M\sum_{k\in I_{n_\beta}} t_{n_\beta}^k \le 2M\alpha_{n_\beta} + \|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}}{1 - t_{n_\beta}}
\end{aligned} \tag{4.8}$$

for $\beta \in D$ and hence

$$\limsup_{\beta\in D}\|u_{n_\beta} - T_1 x\| \le \limsup_{\beta\in D}\|u_{n_\beta} - x\|. \tag{4.9}$$

This is (i). We next show (ii). We assume that $T_1 x = x$. Then $T_1 \circ T_2 x = T_2 \circ T_1 x = T_2 x$. For $\beta \in D$ with $1, 2 \in I_{n_\beta}$, we have

$$\begin{aligned}
\|u_{n_\beta} - T_2 x\| &\le \|u_{n_\beta} - S(I_{n_\beta}, t_{n_\beta})u_{n_\beta}\| + \|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - T_2 x\| \\
&\le \alpha_{n_\beta}\|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - u\| + \left(1 - \sum_{k\in I_{n_\beta}} t_{n_\beta}^k\right)\|T_1 u_{n_\beta} - T_2 x\| \\
&\quad + t_{n_\beta}\|T_2 u_{n_\beta} - T_2 x\| + \sum_{k\in I_{n_\beta}\setminus\{1\}} t_{n_\beta}^k \|T_{k+1}u_{n_\beta} - T_2 x\| \\
&\le 2M\alpha_{n_\beta} + (1 - t_{n_\beta})\|T_1 u_{n_\beta} - T_2 x\| + t_{n_\beta}\|u_{n_\beta} - x\| + 2M\sum_{k\in I_{n_\beta}\setminus\{1\}} t_{n_\beta}^k \\
&\le 2M\alpha_{n_\beta} + (1 - t_{n_\beta})\|T_1 u_{n_\beta} - T_1 \circ T_2 x\| + t_{n_\beta}\|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}^2}{1 - t_{n_\beta}} \\
&\le 2M\alpha_{n_\beta} + (1 - t_{n_\beta})\|u_{n_\beta} - T_2 x\| + t_{n_\beta}\|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}^2}{1 - t_{n_\beta}}
\end{aligned} \tag{4.10}$$

and hence

$$\|u_{n_\beta} - T_2 x\| \le 2M\frac{\alpha_{n_\beta}}{t_{n_\beta}} + \|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}}{1-t_{n_\beta}}. \tag{4.11}$$

Therefore we obtain

$$\limsup_{\beta\in D}\|u_{n_\beta} - T_2 x\| \le \limsup_{\beta\in D}\|u_{n_\beta} - x\|. \tag{4.12}$$

Let us prove (iii). We assume $T_1x = T_2x = \cdots = T_{\ell-1}x = x$ for some $\ell \in \mathbb{N}$ with $\ell \ge 3$. Then $T_m \circ T_\ell x = T_\ell \circ T_m x = T_\ell x$ for every $m \in \mathbb{N}$ with $1 \le m < \ell$. For $\beta \in D$ with $1,2,\ldots,\ell-1 \in I_{n_\beta}$, we have

$$\begin{aligned}
\|u_{n_\beta} - T_\ell x\| &\le \|u_{n_\beta} - S(I_{n_\beta}, t_{n_\beta})u_{n_\beta}\| + \|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - T_\ell x\| \\
&\le \alpha_{n_\beta}\|S(I_{n_\beta}, t_{n_\beta})u_{n_\beta} - u\| + \Bigg(1 - \sum_{k\in I_{n_\beta}} t_{n_\beta}^k\Bigg)\|T_1 u_{n_\beta} - T_\ell x\| \\
&\quad + \sum_{m=1}^{\ell-2} t_{n_\beta}^m \|T_{m+1}u_{n_\beta} - T_\ell x\| + t_{n_\beta}^{\ell-1}\|T_\ell u_{n_\beta} - T_\ell x\| \\
&\quad + \sum_{k\in I_{n_\beta}\setminus\{1,2,\ldots,\ell-1\}} t_{n_\beta}^k \|T_{k+1}u_{n_\beta} - T_\ell x\| \\
&\le 2M\alpha_{n_\beta} + \Bigg(1 - \sum_{m=1}^{\ell-1} t_{n_\beta}^m\Bigg)\|T_1 u_{n_\beta} - T_\ell x\| \\
&\quad + \sum_{m=1}^{\ell-2} t_{n_\beta}^m \|T_{m+1}u_{n_\beta} - T_\ell x\| + t_{n_\beta}^{\ell-1}\|u_{n_\beta} - x\| + 2M\sum_{k\in I_{n_\beta}\setminus\{1,2,\ldots,\ell-1\}} t_{n_\beta}^k \\
&\le 2M\alpha_{n_\beta} + \Bigg(1 - \sum_{m=1}^{\ell-1} t_{n_\beta}^m\Bigg)\|T_1 u_{n_\beta} - T_1\circ T_\ell x\| \\
&\quad + \sum_{m=1}^{\ell-2} t_{n_\beta}^m \|T_{m+1}u_{n_\beta} - T_{m+1}\circ T_\ell x\| + t_{n_\beta}^{\ell-1}\|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}^\ell}{1-t_{n_\beta}} \\
&\le 2M\alpha_{n_\beta} + \Bigg(1 - \sum_{m=1}^{\ell-1} t_{n_\beta}^m\Bigg)\|u_{n_\beta} - T_\ell x\| \\
&\quad + \sum_{m=1}^{\ell-2} t_{n_\beta}^m \|u_{n_\beta} - T_\ell x\| + t_{n_\beta}^{\ell-1}\|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}^\ell}{1-t_{n_\beta}} \\
&= 2M\alpha_{n_\beta} + \Big(1 - t_{n_\beta}^{\ell-1}\Big)\|u_{n_\beta} - T_\ell x\| + t_{n_\beta}^{\ell-1}\|u_{n_\beta} - x\| + 2M\frac{t_{n_\beta}^\ell}{1-t_{n_\beta}}
\end{aligned} \tag{4.13}$$

and hence

$$\|u_{n_\beta} - T_\ell x\| \le 2M \frac{\alpha_{n_\beta}}{t_{n_\beta}^{\ell-1}} + \|u_{n_\beta} - x\| + 2M \frac{t_{n_\beta}}{1 - t_{n_\beta}}. \tag{4.14}$$

Therefore we obtain

$$\limsup_{\beta \in D} \|u_{n_\beta} - T_\ell x\| \le \limsup_{\beta \in D} \|u_{n_\beta} - x\|. \tag{4.15}$$

This completes the proof. □

Remark 4.2. Let g be a strictly increasing mapping on $\mathbb{N}$. Then it is obvious that $\lim_n t_{g(n)} = \lim_n \alpha_{g(n)}/t_{g(n)}^\ell = 0$ for all $\ell \in \mathbb{N}$, $I_{g(n)} \subset I_{g(n+1)}$ for $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty I_{g(n)} = \mathbb{N}$. Thus, the same conclusions of Lemmas 4.3–4.6 also hold for $\{u_{g(n)}\}$.

Lemma 4.3. *Let E, C, $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Assume that $\{u_n\}$ converges strongly to some point $x \in C$. Then x is a common fixed point of $\{T_n : n \in \mathbb{N}\}$.*

Proof. From Lemma 4.1(i), we have

$$\limsup_{n\to\infty} \|u_n - T_1 x\| \le \lim_{n\to\infty} \|u_n - x\| = 0. \tag{4.16}$$

This means $\{u_n\}$ converges to $T_1 x$ and hence $T_1 x = x$. We assume that $T_1 x = \cdots = T_{\ell-1} x = x$ for some $\ell \in \mathbb{N}$ with $\ell \ge 2$. Then from Lemma 4.1(ii) and (iii), we have

$$\limsup_{n\to\infty} \|u_n - T_\ell x\| \le \lim_{n\to\infty} \|u_n - x\| = 0. \tag{4.17}$$

This means $\{u_n\}$ converges to $T_\ell x$ and hence $T_\ell x = x$. So, by induction, we obtain $T_n x = x$ for all $n \in \mathbb{N}$. This completes the proof. □

Lemma 4.4. *Let E, C, $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Assume that E is smooth and $z \in C$ is a common fixed point of $\{T_n : n \in \mathbb{N}\}$. Then*

$$\langle u_n - u, J(u_n - z)\rangle \le 0 \tag{4.18}$$

for all $n \in \mathbb{N}$.

Proof. Since $\alpha_n(u_n - u) = (1 - \alpha_n)(S(I_n, t_n)u_n - u_n)$, we have

$$\begin{aligned}\frac{\alpha_n}{1-\alpha_n}\langle u_n - u, J(u_n - z)\rangle &= \langle S(I_n,t_n)u_n - u_n, J(u_n - z)\rangle \\ &= \langle S(I_n,t_n)u_n - z, J(u_n - z)\rangle + \langle z - u_n, J(u_n - z)\rangle \\ &= \langle S(I_n,t_n)u_n - S(I_n,t_n)z, J(u_n - z)\rangle - \|u_n - z\|^2 \\ &\le \|S(I_n,t_n)u_n - S(I_n,t_n)z\|\,\|u_n - z\| - \|u_n - z\|^2 \\ &\le \|u_n - z\|^2 - \|u_n - z\|^2 = 0.\end{aligned} \tag{4.19}$$

Thus we obtain

$$\langle u_n - u, J(u_n - z) \rangle \le 0 \tag{4.20}$$

for all $n \in \mathbb{N}$. □

LEMMA 4.5. *Let E, C, $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Assume that E is smooth. Then $\{u_n\}$ has at most one cluster point.*

Proof. We assume that a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges strongly to x, and that another subsequence $\{u_{n_j}\}$ of $\{u_n\}$ converges strongly to y. Applying Lemma 4.3 to the subsequences $\{u_{n_i}\}$ and $\{u_{n_j}\}$, we have that x and y are common fixed points of $\{T_n : n \in \mathbb{N}\}$. So, by Lemma 4.4, we have

$$\langle u_{n_i} - u, J(u_{n_i} - y) \rangle \le 0 \tag{4.21}$$

for all $i \in \mathbb{N}$. Therefore we obtain

$$\langle x - u, J(x - y) \rangle \le 0. \tag{4.22}$$

Similarly we can prove

$$\langle y - u, J(y - x) \rangle \le 0. \tag{4.23}$$

So we obtain

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, J(x - y) \rangle \\ &= \langle x - u, J(x - y) \rangle + \langle u - y, J(x - y) \rangle \\ &= \langle x - u, J(x - y) \rangle + \langle y - u, J(y - x) \rangle \le 0. \end{aligned} \tag{4.24}$$

This implies $x = y$. This completes the proof. □

LEMMA 4.6. *Let E be a reflexive Banach space with uniformly Gâteaux differentiable norm and let C be a closed convex subset of E with the fixed point property for nonexpansive mappings. Let $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Then $\{u_n\}$ has a cluster point which is a common fixed point of $\{T_n : n \in \mathbb{N}\}$.*

Proof. From the proof of Lemma 4.1, we have that $\{u_n\}$ is bounded. Take a universal subnet $\{u_\nu : \nu \in D\}$ of $\{u_n\}$. Define a continuous convex function f from C into $[0, \infty)$ by

$$f(x) := \lim_{\nu \in D} ||u_\nu - x|| \tag{4.25}$$

for all $x \in C$. We note that f is well defined because $\{\|u_\nu - x\|\}$ is a universal net in some compact subset of $\mathbb{R}$ for each $x \in C$. From the reflexivity of E and $\lim_{\|x\| \to \infty} f(x) = \infty$, we can put $r := \min_{x \in C} f(x)$ and define a nonempty weakly compact convex subset A of C by

$$A := \{x \in C : f(x) = r\}. \tag{4.26}$$

We will prove that A satisfies the assumption of Theorem 3.2. For each $x \in A$, by Lemma 4.1(i), we have

$$r \le f(T_1 x) = \lim_{\nu \in D} \|u_\nu - T_1 x\| \le \lim_{\nu \in D} \|u_\nu - x\| = f(x) = r \tag{4.27}$$

and hence $T_1 x \in A$. This implies A is T_1-invariant. Fix $x \in A$ with $T_1 x = \cdots = T_\ell x$ for some $\ell \in \mathbb{N}$. Then by Lemma 4.1(ii) and (iii), we have

$$r \le f(T_{\ell+1} x) = \lim_{\nu \in D} \|u_\nu - T_{\ell+1} x\| \le \lim_{\nu \in D} \|u_\nu - x\| = f(x) = r \tag{4.28}$$

and hence $T_{\ell+1} x \in A$. Thus we obtain $T_{\ell+1}(A \cap (\bigcap_{k=1}^{\ell} F(T_k))) \subset A$ for all $\ell \in \mathbb{N}$. So, by Theorem 3.2, there exists a common fixed point z of $\{T_n : n \in \mathbb{N}\}$ in A. We next prove that such z is a cluster point of $\{u_n\}$. By Lemma 4.4, we have

$$\langle u_\nu - u, J(u_\nu - z) \rangle \le 0 \tag{4.29}$$

for all $\nu \in D$. On the other hand, from $z \in A$, we have

$$\lim_{\nu \in D} \langle u - z, J(u_\nu - z) \rangle \le 0 \tag{4.30}$$

by Lemma 2.2. Hence,

$$\begin{aligned} \lim_{\nu \in D} \|u_\nu - z\|^2 &= \lim_{\nu \in D} \langle u_\nu - z, J(u_\nu - z) \rangle \\ &= \lim_{\nu \in D} \langle u_\nu - u, J(u_\nu - z) \rangle + \lim_{\nu \in D} \langle u - z, J(u_\nu - z) \rangle \le 0 \end{aligned} \tag{4.31}$$

holds. Therefore

$$\liminf_{n \to \infty} \|u_n - z\| \le \lim_{\nu \in D} \|u_\nu - z\| = 0, \tag{4.32}$$

that is, z is a cluster point of $\{u_n\}$. This completes the proof. □

Lemma 4.7. *Let E, C, $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Assume that E is smooth. For each $u \in C$, define a sequence $\{Q(u,n)\}$ in C by*

$$Q(u,n) = (1 - \alpha_n) S(I_n, t_n) Q(u,n) + \alpha_n u \tag{4.33}$$

for $n \in \mathbb{N}$. Suppose that for every $u \in C$, $\{Q(u,n)\}$ converges strongly. Then

$$Pu = \lim_{n \to \infty} Q(u,n) \tag{4.34}$$

holds for every $u \in C$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. We put $F(\mathcal{S}) := \bigcap_{n=1}^{\infty} F(T_n)$. Define a mapping P on C by $Pu := \lim_n Q(u,n)$ for $u \in C$. We will prove that such P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$. By Lemma 4.3, we note that $Px \in F(\mathcal{S})$ for all $x \in C$. For $z \in F(\mathcal{S})$, since

$$z = (1 - \alpha_n) S(I_n, t_n) z + \alpha_n z \tag{4.35}$$

for all $n \in \mathbb{N}$, we have $Q(z,n) = z$ for all $n \in \mathbb{N}$. Hence, we obtain $Pz = z$. Therefore we have shown that $P^2 = P$, that is, P is a retraction from C onto $F(\mathcal{S})$. Fix $x \in C$ and $y \in F(\mathcal{S})$. Then, from Lemma 4.4, we have

$$\langle Q(x,n) - x, J(Q(x,n) - y)\rangle \le 0 \tag{4.36}$$

for all $n \in \mathbb{N}$. Since $\{Q(x,n)\}$ converges strongly to Px, we obtain

$$\langle Px - x, J(Px - y)\rangle \le 0. \tag{4.37}$$

So, by Lemma 2.1, such mapping P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$. This completes the proof. □

5. Main results

In this section, we prove our main results. We put $F(\mathcal{S}) := \bigcap_{n=1}^{\infty} F(T_n)$.

Theorem 5.1. *Let E be a reflexive Banach space with uniformly Gâteaux differentiable norm and let C be a closed convex subset of E with the fixed point property for nonexpansive mappings. Let $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.*

Proof. Applying Lemma 4.6 to a subsequence of $\{u_n\}$, we have that every subsequence of $\{u_n\}$ has a cluster point. So, by Lemmas 2.4 and 4.5, we obtain that $\{u_n\}$ converges strongly. So, by Lemma 4.7, we obtain the desired result. □

Theorem 5.2. *Let E be a smooth reflexive Banach space with the Opial property and let C be a closed convex subset of E. Assume that the duality mapping J of E is weakly sequentially continuous at zero. Let $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, $\{I_n\}$, u, and $\{u_n\}$ be as in Lemma 4.1. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.*

Proof. From the proof of Lemma 4.1, we have that $\{u_n\}$ is bounded. Let $\{u_{n_i}\}$ be an arbitrary subsequence of $\{u_n\}$. Since E is reflexive, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to some point $z \in C$. We put $z_j := u_{n_{i_j}}$ for $j \in \mathbb{N}$. Applying Lemma 4.1(i) to $\{z_j\}$, we have

$$\limsup_{j\to\infty} \|z_j - T_1 z\| \le \limsup_{j\to\infty} \|z_j - z\|. \tag{5.1}$$

Since E has the Opial property, we obtain $T_1 z = z$. We assume that $T_1 z = \cdots = T_\ell z = z$ for some $\ell \in \mathbb{N}$. Then, by Lemma 4.1(ii) and (iii), we have

$$\limsup_{j\to\infty} \|z_j - T_{\ell+1} z\| \le \limsup_{j\to\infty} \|z_j - z\|. \tag{5.2}$$

By the Opial property of E again, we obtain $T_{\ell+1}z = z$. Thus, by induction, z is a common fixed point of $\{T_n : n \in \mathbb{N}\}$. By using Lemma 4.4, we have

$$\begin{aligned} \|z_j - z\|^2 &= \langle z_j - z, J(z_j - z)\rangle \\ &= \langle z_j - u, J(z_j - z)\rangle + \langle u - z, J(z_j - z)\rangle \\ &\le \langle u - z, J(z_j - z)\rangle \end{aligned} \tag{5.3}$$

for all $j \in \mathbb{N}$. Since J is weakly sequentially continuous at zero, $\{z_j\}$ converges strongly to z. Hence, $\{u_{n_i}\}$ has a cluster point z. So, by Lemmas 2.4 and 4.5, $\{u_n\}$ itself converges strongly. Thus, by Lemma 4.7, we obtain the desired result. □

Remark 5.3. In Theorems 5.1 and 5.2, from the proofs of Lemma 4.6 and Theorem 5.2, we may replace the condition of the reflexivity of E by the weaker condition that C is locally weakly compact.

By Theorems 5.1 and 5.2, we obtain the following.

THEOREM 5.4. *Let C be a weakly compact convex subset of a Banach space E. Assume that either of the following holds:*

(i) *E is uniformly smooth; or*

(ii) *E is a smooth Banach space with the Opial property and the duality mapping J of E is weakly sequentially continuous at zero.*

Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0,1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n/t_n^\ell = 0$ for $\ell \in \mathbb{N}$. Let $\{I_n\}$ be a sequence of nonempty subsets of $\mathbb{N}$ such that $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty I_n = \mathbb{N}$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1-\alpha_n)\left(\left(1 - \sum_{k\in I_n} t_n^k\right)T_1u_n + \sum_{k\in I_n} t_n^k T_{k+1}u_n\right) + \alpha_n u \tag{5.4}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Remark 5.5. By Theorem 3.1, we know $F(\mathcal{S}) \neq \emptyset$.

Example 5.6. Define sequences $\{\alpha_n\}$ and $\{t_n\}$ by $\alpha_n := 1/n^n$ and $t_n := 1/n$ for $n \in \mathbb{N}$. Then $\{\alpha_n\}$ and $\{t_n\}$ satisfy $\lim_n t_n = \lim_n \alpha_n/t_n^\ell = 0$ for $\ell \in \mathbb{N}$.

COROLLARY 5.7. *Let E, C, $\{T_n\}$, $\{\alpha_n\}$, $\{t_n\}$, and P be as in Theorem 5.4. Fix $u \in C$ and define sequences $\{u_n\}$ and $\{v_n\}$ in C by*

$$\begin{aligned} u_n &= (1-\alpha_n)\left(\left(1 - \sum_{k=1}^{n} t_n^k\right)T_1u_n + \sum_{k=1}^{n} t_n^k T_{k+1}u_n\right) + \alpha_n u, \\ v_n &= (1-\alpha_n)\left(\left(1 - \sum_{k=1}^{\infty} t_n^k\right)T_1v_n + \sum_{k=1}^{\infty} t_n^k T_{k+1}v_n\right) + \alpha_n u \end{aligned} \tag{5.5}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ and $\{v_n\}$ converge strongly to Pu.

From the proofs of lemmas in Section 4, we also obtain the following.

Theorem 5.8. *Let E and C be as in Theorem 5.4. Let $\{T_n : n = 1,2,\dots,\ell\}$ be a finite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0,1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n/t_n^{\ell-1} = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1-\alpha_n)\left(\left(1-\sum_{k=1}^{\ell-1} t_n^k\right)T_1u_n + \sum_{k=1}^{\ell-1} t_n^k T_{k+1}u_n\right) + \alpha_n u \tag{5.6}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{k=1}^{\ell} F(T_k)$.

6. ℓ-parameter nonexpansive semigroups

In this section, we apply Theorem 5.8 to ℓ-parameter nonexpansive semigroups. We recall that a family of mappings $\{T(p) : p \in [0,\infty)^\ell\}$ is said to be an *ℓ-parameter nonexpansive semigroup* on a closed convex subset C of a Banach space E if the following are satisfied.

(i) For each $p \in [0,\infty)^\ell$, $T(p)$ is a nonexpansive mapping on C.

(ii) $T(p+q) = T(p) \circ T(q)$ for all $p,q \in [0,\infty)^\ell$.

(iii) For each $x \in C$, the mapping $p \mapsto T(p)x$ from $[0,\infty)^\ell$ into C is continuous.

The following is proved in [22]. See also [23].

Theorem 6.1 [22]. *Let $\{T(p) : p \in [0,\infty)^\ell\}$ be an ℓ-parameter nonexpansive semigroup on a closed convex subset C of a Banach space E. Let $p_1,p_2,\dots,p_\ell \in [0,\infty)^\ell$ such that $\{p_1,p_2,\dots,p_\ell\}$ is linearly independent in the usual sense. Let $\beta_1,\beta_2,\dots,\beta_\ell \in \mathbb{R}$ such that $\{1,\beta_1,\beta_2,\dots,\beta_\ell\}$ is linearly independent over $\mathbb{Q}$, that is,*

$$\nu_0 + \nu_1\beta_1 + \nu_2\beta_2 + \cdots + \nu_\ell\beta_\ell = 0 \quad \text{implies } \nu_0 = \nu_1 = \nu_2 = \cdots = \nu_\ell = 0 \tag{6.1}$$

for $\nu_0,\nu_1,\nu_2,\dots,\nu_\ell \in \mathbb{Z}$. Suppose $p_0 := \beta_1p_1 + \beta_2p_2 + \cdots + \beta_\ell p_\ell \in [0,\infty)^\ell$. Then

$$\bigcap_{p\in[0,\infty)^\ell} F(T(p)) = F(T(p_0)) \cap F(T(p_1)) \cap F(T(p_2)) \cap \cdots \cap F(T(p_\ell)) \tag{6.2}$$

holds.

By Theorems 5.8 and 6.1, we obtain the following.

Theorem 6.2. *Let E and C be as in Theorem 5.4. Let $\{T(p)\}$, $\{p_0,p_1,p_2,\dots,p_\ell\}$, $\{\beta_1,\beta_2,\dots,\beta_\ell\}$ be as in Theorem 6.1. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0,1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n/t_n^\ell = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1-\alpha_n)\left(\left(1-\sum_{k=1}^{\ell} t_n^k\right)T(p_0)u_n + \sum_{k=1}^{\ell} t_n^k T(p_k)u_n\right) + \alpha_n u \tag{6.3}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{p\in[0,\infty)^\ell} F(T(p))$.

When $\ell = 1$, Theorem 6.2 becomes the following, which differs from Theorem 1.2.

Corollary 6.3. *Let E and C be as in Theorem 5.4. Let* $\{T(t) : t \geq 0\}$ *be a one-parameter nonexpansive semigroup on C. Let* $\{\alpha_n\}$ *and* $\{t_n\}$ *be sequences in* $(0, 1/2)$ *satisfying* $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. *Let* σ *and* τ *be positive real numbers satisfying* $\sigma/\tau \notin \mathbb{Q}$. *Fix* $u \in C$ *and define sequences* $\{u_n\}$ *and* $\{v_n\}$ *in C by*

$$\begin{aligned} u_n &= (1-\alpha_n)\big((1-t_n)T(\sigma)u_n + t_n T(\tau)u_n\big) + \alpha_n u, \\ v_n &= (1-t_n-\alpha_n)T(\sigma)v_n + t_n T(\tau)v_n + \alpha_n u \end{aligned} \tag{6.4}$$

for $n \in \mathbb{N}$. *Then* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto* $\bigcap_{t \geq 0} F(T(t))$.

Proof. We remark that

$$\begin{aligned} v_n &= (1-\alpha_n)\left(\left(1 - \frac{t_n}{1-\alpha_n}\right)T(\sigma)v_n + \frac{t_n}{1-\alpha_n}T(\tau)v_n\right) + \alpha_n u, \\ \lim_{n\to\infty} \frac{\alpha_n}{t_n/(1-\alpha_n)} &= \lim_{n\to\infty} \frac{\alpha_n(1-\alpha_n)}{t_n} = 0. \end{aligned} \tag{6.5}$$

From this thing, we can obtain the desired result. □

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